Lecture 7

Stochastic MPC III
Course outline

1. Classical (nominal) MPC:
   ▶ Dual mode prediction paradigm, recursive feasibility and stability
   ▶ Constraint handling, controller dynamics, computation and implementation

2. Robust MPC
   ▶ Additive model uncertainty – tube MPC; open loop and closed loop prediction strategies.
   ▶ Parametric model uncertainty – controller dynamics, recursive state bounding, polytopic tubes.

3. Stochastic MPC
   ▶ Probabilistic constraints, recursive feasibility, stability and convergence.
   ▶ Parametric uncertainty, state decomposition, output feedback
     ▶ Sample-based methods; polytopic tubes revisited

4. Online dynamic programming
   ▶ Active set method for additive model uncertainty; $H_\infty$ cost
   ▶ Parametric uncertainty; controllable set computation

Reading


Sample approximations of chance-constrained programs:
Confidence bounds and clustering

Optimization with probabilistic constraints

Consider optimization problems involving random variables that must satisfy constraints with a specified minimum probability

Probabilistic (or chance) constraints allow better optimality and greater parametric uncertainty than robust constraints

Applications are numerous and diverse:
... finance, portfolio management, production planning, supply chain management, sustainable development, chemical process design, telecommunications networks, building control...

Stochastic Programming methods have been developed since the 1950's e.g. Charnes & Cooper, Management Sci., 1959 but exact handling of probabilistic constraints remains mostly intractable
Define the chance constrained problem

\[
\text{(CCP)}: \quad \begin{array}{l}
\text{minimize} \quad f(x) \\
\text{subject to} \quad \mathbb{P}\{\delta \in \Delta : g(x, \delta) \leq 0\} \geq p.
\end{array}
\]

for scalar functions \(f, g\), and compact domain \(\mathcal{D}\)

\(\mathbb{P}\{S\} = \text{probability that a realization of the random variable} \ \delta \in \Delta \ \text{lies in} \ S\)

Assume:

\begin{itemize}
\item \(g(x, \delta)\) is upper semicontinuous in \(x \in \mathcal{D}\) for each \(\delta \in \Delta\)
\item \(g(x, \delta)\) is lower semicontinuous in \(\delta \in \Delta\) for each \(x \in \mathcal{D}\)
\item \(\text{(CCP)}\) is feasible
\end{itemize}

i.e. \(\mathbb{P}\{\delta \in \Delta : g(x, \delta) \leq 0\} \geq p\) for some \(x \in \mathcal{D}\).

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Optimization with probabilistic constraints

Two major difficulties with optimization subject to probabilistic constraints:

1. Prohibitive computation to determine feasibility of given \(x\)

\[
\mathbb{P}\{\delta \in \Delta : g(x, \delta) \leq 0\} = \int_{\delta \in \Delta} \mathbb{1}_{\{g(x, \delta) \leq 0\}}(\delta) \ p(\delta) \ d\delta
\]

- Closed form expression only available in special cases, e.g.
  \[
\begin{align*}
\delta &\in \mathcal{N}(0, I) \\
g(x, \delta) &= c^T \delta - 1 + (d + D^T \delta) \ T \ x
\end{align*}
\]
  Gaussian uncertainty
  jointly affine
gives

\[
\mathbb{P}\{\delta \in \Delta : g(x, \delta) \leq 0\} \geq p \iff d^T x - 1 \geq \Phi^{-1}(p) \|c + Dx\|_2
\]

- In general multidimensional integration is needed
  e.g. using Monte Carlo methods
Optimization with probabilistic constraints

Two major difficulties with optimization subject to probabilistic constraints:

2. The feasible set $\mathcal{F}$ may be nonconvex even if $g(x, \delta)$ is convex in $x$

\[ e.g. \ d^T x - 1 \geq \Phi^{-1}(p) \| c + D x \|_2 \text{ is convex if } p \geq 0.5 \]
\[ \text{nonconvex if } p < 0.5 \]

\[ F_{p > 0.5} \]
\[ F_{p < 0.5} \]

Sample approximation

A sample of $N$ independent and identically distributed realizations of $\delta$

\[ \omega := \{ \delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)} \} \]

allows an approximation

\[ \mathbb{P}\{ \delta \in \Delta : g(x, \delta) \leq 0 \} = \int_{\delta \in \Delta} \mathbb{1}_{\{ g(x, \delta) \leq 0 \}}(\delta) \ p(\delta) \ d\delta \]
\[ \approx \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{\{ g(x, \delta) \leq 0 \}}(\delta^{(j)}) \]

* This approximation is relatively easy to evaluate

* The approximation error is a random variable since $\omega$ is a random variable but it is likely to decrease as $N$ increases
Sample approximation

The sample-counterpart of (CCP) is

\[ \text{(SP)} : \min_{x \in \mathcal{D}} f(x) \]

s.t. \( g(x, \delta^{(j)}) \leq 0 \) for all \( j \in \mathcal{I} \subseteq \{1, \ldots, N\} \)

\(|\mathcal{I}| \geq q\)

for some chosen \( q \), with \( n_x \leq q \leq N \)

Assume:

- The sample \( \{\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)}\} \) independent and identically distributed according to \( \mathbb{P} \)

- (SP) is feasible
  
  i.e. \( g(x, \delta^{(j)}) \leq 0 \) for all \( j \in \mathcal{I} \), with \( |\mathcal{I}| \geq q \), for some \( x \in \mathcal{D} \)

  (otherwise consider all probabilities to be conditional on feasibility of (SP))

Sample approximation

Solutions of (SP) converge to the solution of (CCP) as \( N \to \infty \) if \( q(N) \to pN \)

but computation depends heavily on \( N \),

so we need to know how approximation accuracy depends on \( N \)

Approximation accuracy can be characterized in terms of bounds on:

- the probability that a solution of (SP) is feasible for (CCP), and

- the probability that the optimal value of (SP) exceeds that of (CCP)


Calafiore, *SIAM J. Optim.*, 2010

Sample approximation

Values of $N$ such that a solution of (SP) is feasible for (CCP) with a confidence of $1 - 10^{-10}$ (for $p = 0.9$, $n_x = 5$)  

Campi & Garatti, 2011

For real time control with sample rate $> 0.1$ Hz, require $N \lesssim 500$

Sample approximation

Using the confidence bounds derived in Campi & Garatti, 2011:

For $N = 500$, $q = 375$: (SP) solves (CCP) with $p = 0.65$ with confidence $F > 1 - 10^{-4}$

but $p_{0.95} - p_{0.05} = 0.125$ suggests a significant degree of conservativeness
Analysis of sample approximation accuracy

Earlier work considered the feasible set for \((SP)\) as the intersection of \(q\) sets
\[
\mathcal{F}_x(\delta^{(j)}) := \{x : g(x, \delta^{(j)}) \leq 0\}
\]

New approach

(i). For given \(x\):

* consider the set
\[
\mathcal{F}_\delta(x) := \{\delta \in \Delta : g(x, \delta) \leq 0\}
\]
* Identify \(S(x) \subset \Delta\), such that
\[
\mathbb{P}\{S(x)\} = p \quad \text{and} \quad S(x) \subseteq \mathcal{F}_\delta(x) \quad \text{or} \quad \mathcal{F}_\delta(x) \subset S(x)
\]

(ii). Considering \(x\) as a random variable, bound the probability that
\[
S(x) \subseteq \mathcal{F}_\delta(x)
\]
Level set analysis

Define \( S_\alpha(x) \subseteq \Delta \), for given \( x \in D \) and \( \alpha \in \mathbb{R} \), as the sublevel set

\[
S_\alpha(x) := \{ \delta \in \Delta : g(x, \delta) \leq \alpha \}
\]

then \( F_\delta(x) = S_0(x) \), and

\[
\begin{aligned}
&\star S_\alpha(x) \subseteq S_0(x) \text{ if and only if } \alpha < 0 \\
&\star S_\alpha(x) \supseteq S_0(x) \text{ if and only if } \alpha \geq 0
\end{aligned}
\]

follows from \( g(x, \delta) \) lower semicontinuous in \( \delta \in \Delta \)

Define the function \( \alpha_p : D \to \mathbb{R} \) for given \( p \in (0, 1] \) as

\[
\alpha_p(x) := \min_\alpha \text{ s.t. } \mathbb{P}\{S_\alpha(x)\} \geq p
\]

to simplify notation we use \( S_{\alpha_p}(x) \) to denote \( S_{\alpha_p}(x)(x) \)

Level set analysis

Proposition \( \alpha_p(x) \) is finite for all \( x \in D \)

Proof: \( \alpha_p(x) = \min_\alpha \{ \alpha \text{ s.t. } \mathbb{P}\{S_\alpha(x)\} \geq p \} \) implies lower and upper bounds:

\[
\begin{aligned}
&\downarrow \alpha_p(x) \geq \max_\delta \in S_{\alpha_p} g(x, \delta) \\
&\text{but } p > 0 \Rightarrow S_{\alpha_p} \cap S' \neq \emptyset \text{ for some compact set } S' \subseteq \Delta \text{ so}
\end{aligned}
\]

\[
S_{\alpha_p} \cap S' \neq \emptyset \Rightarrow \alpha_p(x) \geq \max_\delta \in S_{\alpha_p} g(x, \delta) \geq \min_\delta \in S' g(x, \delta) > -\infty
\]

(since \( g(x, \delta) \) is lower-semicontinuous in \( \delta \))

\[
\downarrow \text{ Feasibility of (CCP) implies } \mathbb{P}\{S_0(x_0)\} \geq p \text{ for some } x_0 \in D
\]

also \( g(x, \delta) \) is finite \( \forall (x, \delta) \in D \times S_0(x_0) \) hence

\[
\mathbb{P}\{S_0(x_0)\} \geq p \quad \Rightarrow \quad \alpha_p(x) \leq \max_{\delta \in S_0(x_0)} g(x, \delta) < +\infty
\]

(since \( D \) is compact and \( g(x, \delta) \) upper-semicontinuous in \( x \))
Level set analysis

If $x$ is feasible for (SP), then
\[ \delta(j) \in S_0(x) \quad \text{for all } j \in \mathcal{I} \subseteq \{1, \ldots, N\}, \ |\mathcal{I}| \geq q \]
\[ \delta(j) \notin S_0(x) \quad \text{for all } j \in \{1, \ldots, N\} \setminus \mathcal{I} \]

Define $\bar{\mathcal{I}}$:
\[ \delta(j) \in \partial S_0(x) \quad \text{for all } j \in \bar{\mathcal{I}} \subseteq \mathcal{I} \]
\[ \delta(j) \in \text{int}(S_0(x)) \quad \text{for all } j \in \mathcal{I} \setminus \bar{\mathcal{I}} \]

Define $\mathcal{I}_p$:
\[ \delta(j) \in S_{\alpha_p}(x) \quad \text{for all } j \in \mathcal{I}_p \subseteq \{1, \ldots, N\} \]
\[ \delta(j) \notin S_{\alpha_p}(x) \quad \text{for all } j \in \{1, \ldots, N\} \setminus \mathcal{I}_p \]

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Level set analysis

The number of indices in $\mathcal{I}_p$ determines whether $\alpha_p$ is positive or negative

**Proposition** For given $x \in \mathcal{D}$ and $p \in (0, 1]$

\[ S_{\alpha_p}(x) \subset S_0(x) \iff \alpha_p(x) < 0 \quad \text{iff } |\mathcal{I}_p| \leq |\mathcal{I}| - |\bar{\mathcal{I}}| \quad (A) \]
\[ S_{\alpha_p}(x) \supseteq S_0(x) \iff \alpha_p(x) \geq 0 \quad \text{iff } |\mathcal{I}_p| \geq |\mathcal{I}| \quad (B) \]
\[ S_{\alpha_p}(x) \subset S_0(x) \iff \alpha_p(x) < 0 \quad \text{if } |\mathcal{I}_p| \leq q - |\bar{\mathcal{I}}| \quad (C) \]

**Proof:** Follows from
\begin{itemize}
  \item for all $x \in \mathcal{D}$, either $S_{\alpha_p}(x) \subset S_0(x)$ or $S_{\alpha_p}(x) \supseteq S_0(x)$
  \item $\mathcal{I}$, $\bar{\mathcal{I}}$ and $\mathcal{I}_p$ are subsets of $\{1, \ldots, N\}$
\end{itemize}
Level set analysis

The number of indices in $I_p$ determines whether $\alpha_p$ is positive or negative

**Proposition** For given $x \in \mathcal{D}$ and $p \in (0, 1]$

1. $S_{\alpha_p}(x) \subset S_0(x) \iff \alpha_p(x) < 0$ iff $|I_p| \leq |I| - |\bar{I}|$ (A)
2. $S_{\alpha_p}(x) \supseteq S_0(x) \iff \alpha_p(x) \geq 0$ iff $|I_p| \geq |I|$ (B)
3. $S_{\alpha_p}(x) \subset S_0(x) \iff \alpha_p(x) < 0$ if $|I_p| \leq q - |\bar{I}|$ (C)

**Proof:**

(A): $\exists \delta^{(j)} \in S_{\alpha_p}(x) \setminus S_0(x)$

\[ |I_p| \leq |I| - |\bar{I}| \]

(B): $\forall \delta^{(j)} \in S_0(x)$, $\delta^{(j)} \in S_{\alpha_p}(x)$

\[ |I_p| \geq |I| \]

Confidence bounds on approximation accuracy

The solution $x$ of (SP) depends on

$\omega = \{\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)}\} \in \Delta^N$

which is a random variable with probability measure $\mathbb{P}^N$

Hence use $\mathbb{P}^N$ to bound the probabilities of (A), (B), (C)

Assume:

(a). The solution of (SP) is unique or criteria are available to select between solutions achieving the same optimal cost

(b). $|\bar{I}| \leq r \leq n_x$ for all $\omega \in \Delta^N$

Here (b) is equivalent to the assumption that there are no weakly active constraints at the solution of (SP)

(note that weakly active constraints occur on a set of zero measure in $\Delta^N$)
Confidence bounds on approximation accuracy

Let $F_{N,n}(p)$ denote the binomial distribution function:

$$F_{N,n}(p) := \sum_{i=0}^{n} \binom{N}{i} p^i (1-p)^{N-i}$$

$F_{N,n}(p)$ is the probability of $n$ or fewer successes in $N$ independent trials, each with probability $p$ of success.

Therefore the probability of $n$ or fewer samples $\delta^{(j)} \in \omega$ lying in a set of measure $p$ is

$$\mathbb{P}^N\{|I_p| \leq n\} = F_{N,n}(p)$$

Confidence bounds on approximation accuracy

**Theorem**  The solution $x$ of (SP) satisfies

$$\mathbb{P}^N\{\mathbb{P}\{S_0(x)\} > p\} \geq F_{N,q-r}(p)$$

and, for $|I| = q$,

$$\mathbb{P}^N\{\mathbb{P}\{S_0(x)\} > p\} \leq F_{N,q-1}(p)$$

**Proof:**

- $S_{\alpha_p}(x) \subset S_0(x)$ if $|I_p| \leq q - r$, so

$$\mathbb{P}^N\{\mathbb{P}\{S_0(x)\} > p\} \geq \mathbb{P}^N\{|I_p| \leq q - r\} = F_{N,q-r}(p)$$

- if $|I| = q$, then $|I_p| < q$ whenever $S_{\alpha_p}(x) \subset S_0(x)$, so

$$\mathbb{P}^N\{\mathbb{P}\{S_0(x)\} > p\} \leq \mathbb{P}^N\{|I_p| < q\} = F_{N,q-1}(p)$$
Confidence bounds on approximation accuracy

Corollary If the solution $x$ of (SP) satisfies $|\mathcal{I}| = q$ and $|\tilde{\mathcal{I}}| = r$, then

$$\mathbb{P}^N\{\mathbb{P}\{\mathcal{S}_0(x)\} > p\} = F_{N,q-r}(p)$$

Proof: In this case $\mathcal{S}_{\alpha_p}(x) \subset \mathcal{S}_0(x)$ iff $|\mathcal{I}_p| \leq q - r$

Note: this applies to problems (SP) for which the number of support constraints is $r$ for all $\omega \in \Delta^N$

c.f. fully supported problems with $r = n_x$

Campi & Garatti, 2010

Confidence bounds on approximation accuracy

Consider the relationship between optimal costs: $J^*(p)$ of (CCP) and $J^*_{N,q}(\omega)$ of (SP)

Corollary

$$\mathbb{P}^N\{\omega \in \Delta^N : J^*_{N,q}(\omega) \geq J^*(p)\} \geq F_{N,q-r}(p)$$

Proof: $J^*_{N,q}(\omega) \geq J^*(p)$ if the solution $x$ of (SP) is feasible for (CCP), so

$$\mathbb{P}^N\{\omega \in \Delta^N : J^*_{N,q}(\omega) \geq J^*(p)\} \geq \mathbb{P}^N\{\mathbb{P}\{\mathcal{S}_0(x)\} > p\}$$

Corollary

$$\mathbb{P}^N\{\omega \in \Delta^N : J^*_{N,q}(\omega) \leq J^*(p)\} \leq F_{N,q-1}(p)$$

Proof: $J^*_{N,q}(\omega) \leq J^*(p)$ if the solution $\hat{x}$ of (CCP) is feasible for (SP) i.e. if $\mathcal{S}_0(\hat{x})$ contains $q$ or more samples $\delta^{(j)}$

so

$$\mathbb{P}^N\{\omega \in \Delta^N : J^*_{N,q}(\omega) \leq J^*(p)\} \geq 1 - F_{N,q-1}(p)$$
Confidence bounds on approximation accuracy

Comparison with previous results:

- Calafiore (2010) and Campi & Garatti (2011) give confidence bounds:
  \[ \mathbb{P}^N \{ \omega \in \Delta^N : \mathbb{P} \{ S_0(x(\omega)) \} > p \} \geq \tilde{F}_{N,q,r}(p) \]
  \[ \mathbb{P}^N \{ \omega \in \Delta^N : J^*_N,q(\omega) \geq J^*(p) \} \geq \tilde{F}_{N,q,r}(p) \]

  where \( \tilde{F}_{N,q,r}(p) := \max \left\{ 0, 1 - \left( \frac{N-q+r-1}{N-q} \right) (1 - F_{N,q-r}(p)) \right\} \)

  These are more conservative since for any given \( N, q \) and \( r \):
  \[ F_{N,q-r}(p) \geq \tilde{F}_{N,q,r}(p) \quad \text{for all } p \in (0, 1] \]
  with equality only if \( r = 1 \)

- Calafiore (2010) and Campi & Garatti (2011) assume
  \* \( g(x, \delta) \) is lower-semi continuous in \( x \in \mathcal{D} \) for each \( \delta \in \Delta \)
  \* \( f, g \) are convex in \( x \)

  Calafiore (2010) also considers the case that (SP) is infeasible

Confidence bounds on approximation accuracy - example

Confidence bounds \( F_{N,q-1}(p), F_{N,q-r}(p) \) and \( \tilde{F}_{N,q,r}(p) \)
for \( N = 500 \) samples with \( q = \lceil 0.75N \rceil = 375 \) and \( r = 3 \)

![Graph showing confidence bounds](image-url)
Confidence bounds on approximation accuracy - example

Values of $p$ lying on 5% and 95% confidence bounds for varying $N$ with $q = [0.75N]$, $r = 3$

![Graph showing confidence bounds for $p$ vs $N$]

(a). convergence:

$$p_{u,5} - p_{l,95} \sim O(N^{-0.5})$$

consistent with Central Limit Theorem

(b). $p_{u,5} - \tilde{p}_{l,95} \approx 2(p_{u,5} - p_{l,95})$

for $N \geq 500$

(a) & (b) imply $\approx 4$ times larger $N$ is needed for same confidence when $\tilde{F}$ used instead of $F$
MIP implementation

Represent (SP) as a Mixed Integer Program (MIP)

\[
\text{(SP-B)}: \quad \min_{x \in \mathcal{D}} f(x) \quad \text{s.t.} \quad g(x, \delta^{(j)}) \leq (1 - b^{(j)})M \\
\quad \quad \quad \quad \quad \quad b^{(j)} \in \{0, 1\}, \quad j = 1, \ldots, N \\
\quad \quad \quad \quad \quad \quad \sum_{j=1}^{N} b^{(j)} \geq q
\]

for some scalar \( M \gg 1 \)

• If \( g(x, \delta^{(j)}) \leq M \) for all \( x \in \mathcal{D} \) and all \( j \)
  then (SP-B) is equivalent to (SP)

• Number of possible values for the set of binary variables \( \{b^{(1)}, \ldots, b^{(N)}\} \) increases exponentially with \( N \)
  hence computation (e.g. branch & bound) depends exponentially on sample size \( N \)

Sample clustering

Reduce computational load of (SP) by pre-processing the sample set \( \omega \), using e.g. k-means clustering

Approach is devised for a sequence of parameterized problems with

\[
\begin{align*}
\text{objective} & \quad f(x; y_t) \\
\text{constraint functions:} & \quad g(x, \delta; y_t) \quad \text{for } t = 0, 1, \ldots
\end{align*}
\]

where the parameter \( y_t \) becomes known at time \( t \)

e.g. in Stochastic MPC:

• \( y_t \) is the state of the controlled system at time \( t \)

• pre-processing is performed offline to speed up online optimization at \( t = 0, 1, \ldots \)
Sample clustering

Clustering reduces the number of binary variables while preserving information on the distribution characterized by $\omega$.

Define $N_c$ clusters: $\mathcal{I}^{(k)} \subseteq \{1, \ldots, N\}$, $k = 1, \ldots, N_c$, with

$$\bigcup_{k=1}^{N_c} \mathcal{I}^{(k)} = \{1, \ldots, N\} \text{ and } \mathcal{I}^{(k)} \cap \mathcal{I}^{(l)} = \emptyset \text{ for all } k \neq l$$

Reformulate (SP-B) as

(SP-C): \[ \min_{x \in \mathcal{D}} f(x) \text{ s.t. } g(x, \delta^{(j)}) \leq (1 - b^{(k)})M \text{ for all } j \in \mathcal{I}^{(k)}, \]

\[ b^{(k)} \in \{0, 1\}, \ k = 1, \ldots, N_c \]

\[ \sum_{k=1}^{N_c} b^{(k)}|\mathcal{I}^{(k)}| \geq q \]

- Various clustering algorithms available (e.g. k-means)
- All samples in a cluster are activated simultaneously, so (SP-C) has $N_c < N$ binary var.s but same number of inequality constraints as (SP-B)

Sample clustering

Let $x(\omega)$ denote a solution of (SP-C)

then:

- $x$ is feasible for (CCP) with probability at least $\mathcal{F}_{N,q-r}(p)$
- $x$ is feasible for (SP-C) with probability no greater than $\mathcal{F}_{N,q-r}(p)$ if $|\mathcal{I}| = q$, $\mathcal{I} = \{j \in \{1, \ldots, N\} : g(x, \delta^{(j)}) \leq 0\}$

Let $J_{N,q,N_c}(\omega)$ denote the optimal objective of (SP-C)

then:

- $\mathbb{P}^N \{ \omega \in \Delta^N : J_{N,q,N_c}(\omega) \geq J^*(p) \} \geq F_{N,q-r}(p)$
- $\mathbb{P}^N \{ \omega \in \Delta^N : J_{N,q,N_c}(\omega) \geq J^*(p) \} \leq F_{N,q-1}(p)$

Hence (SP-C) has the same confidence bounds as (SP)
Example

Linear objective function: \( f(x) = f^T x, \quad f^T = [-0.90, -0.56, 0.517] \)

Piecewise linear constraint function:
\[
g(x, \delta) = \max_i c_i^T \delta + (d_i + D_i \delta)^T x - 1
\]
\[
c_1 = \begin{bmatrix} -0.33 \\ -0.68 \end{bmatrix}, \quad d_1 = \begin{bmatrix} -0.20 \\ -0.39 \\ 0.04 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.35 & 0.25 \\ 0.90 & -0.65 \\ 0.89 & -0.83 \end{bmatrix}
\]
\[
c_2 = \begin{bmatrix} 0.88 \\ 0.74 \end{bmatrix}, \quad d_2 = \begin{bmatrix} -0.65 \\ 0.51 \\ -0.30 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -0.58 & 0.72 \\ -0.43 & -0.59 \\ -0.32 & 0.28 \end{bmatrix}
\]
\[
c_3 = \begin{bmatrix} -0.75 \\ 0.68 \end{bmatrix}, \quad d_3 = \begin{bmatrix} -0.84 \\ 0.55 \\ -0.77 \end{bmatrix}, \quad D_3 = \begin{bmatrix} -0.68 & -0.05 \\ 0.81 & 0.41 \\ 0.58 & -0.22 \end{bmatrix}
\]
\[
c_4 = \begin{bmatrix} -0.19 \\ -0.93 \end{bmatrix}, \quad d_4 = \begin{bmatrix} -0.61 \\ -0.94 \\ 0.57 \end{bmatrix}, \quad D_4 = \begin{bmatrix} -0.76 & 0.74 \\ 0.68 & 0.52 \\ 0.99 & 0.46 \end{bmatrix}
\]

Gaussian uncertainty: \( \delta \sim \mathcal{N} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1.21 & -0.24 \\ -0.24 & 0.75 \end{bmatrix} \right) \)

Domain: \( D = \{ x \in \mathbb{R}^3 : \| x \|_{\infty} \leq 10^3 \} \)

Confidence bounds and observed distribution of \( \mathbb{P}\{ S_0(x) > p \} \), computed by:
(a). solving (SP-B) \( 10^3 \) times as a MIP
(b). using \( 10^5 \) samples to compute \( \mathbb{P}\{ S_0(x) \} \) empirically for each solution \( x \)

Example: Confidence bounds

For this example \( |\hat{I}| = n_x = 3 \) for all \( \omega \)
so the observed distributions of \( \mathbb{P}\{ S_0(x) \} > p \) lie approximately on \( F_{N,q-3}(p) \)
Example: Effect of clustering on computational load

Comparison of computational load with and without clustering –
average execution times (using Gurobi, 2.3 GHz i7 quad core processor):

<table>
<thead>
<tr>
<th>$N (N_c)$</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>200 (100)</th>
<th>500 (100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>time (s)</td>
<td>0.37</td>
<td>1.38</td>
<td>30.0</td>
<td>0.73</td>
<td>1.49</td>
</tr>
</tbody>
</table>

- Computation for $N = 200, 500$ and $N_c = 100$ is comparable to $N = 100$, but (slightly) higher due to greater number of inequality constraints
- For $N = 500$ and no clusters, the solver terminated before converging to the optimum in almost all cases

Example: Confidence bounds with clustering

Observed distributions of $\mathbb{P}\{S_0(x)\} > p$ with $N_c = 100$ clusters for $N = 500$, the upper confidence bound is exceeded because $|T| > q$
Typical distribution of samples and clusters

Solid lines: boundaries of $S_0(x)$ for $N = 500$, $N_c = 0$
Dashed lines: boundaries of $S_0(x)$ for $N = 500$, $N_c = 100$
Consider LPV systems with polytopic multiplicative uncertainty

$$x_{k+1} = A_k x_k + B_k u_k, \quad (A_k, B_k) \in \Theta, \quad k = 0, 1, \ldots$$

where the probability distribution of $(A_k, B_k)$ is known and

$$\Theta := \text{Co}\{(A^{(j)}, B^{(j)}), \quad j = 1, \ldots, p\}$$

The aim is to minimizes a multistage cost

$$J(x_0, u) = \sum_{k=0}^{\infty} L(x_k, u_k)$$

where $J(x_0, u)$ is positive definite in $x_0$ and $u$

e.g. an expected value (or nominal) quadratic cost

subject to hard constraints and probabilistic constraints:

$$F_h x_k + G_h u_k \leq 1 \quad k = 0, 1, \ldots$$
$$\Pr_{x_k}(F_s x_{k+1} + G_s u_{k+1} \leq 1) \geq p \quad k = 0, 1, \ldots$$

Let $x_{i|k}$, $u_{i|k}$ be the predicted values at time $k$ of $x_{k+i}$, $u_{k+i}$

The input sequence $\{u_{0|k}, u_{1|k}, \ldots\}$ predicted at time $k$ is parameterized as

$$u_{i|k} = K x_{i|k} + c_{i|k}$$

where $u = K x$ is stabilizing in the absence of constraints and

$$c_k = \{c_{0|k}, \ldots, c_{N-1|k}\} \text{ are decision variables at time } k$$
$$c_{i|k} = 0 \text{ for all } i \geq N$$

for a given horizon $N$

The predicted state sequence $\{x_{0|k}, x_{1|k}, \ldots\}$ evolves as

$$x_{i+1|k} = \Phi_k x_{i|k} + B_k c_{i|k}$$

for $\Phi_k = A_k + B_k K \in \text{Co}\{\Phi^{(j)}, \quad j = 1, \ldots, p\}$,
$$\Phi^{(j)} = A^{(j)} + B^{(j)} K$$
Stochastic MPC

Model uncertainty is handled by determining a tube \( \{ X_{i|k}, i = 0, 1, \ldots \} \) such that \( x_{i|k} \in X_{i|k} \) for all \( i \), for all realizations of model uncertainty.

Multiplicative uncertainty implies \( u_{i|k} \) affects the influence of uncertainty on predicted states \( x_{l|k}, l > i \)

\( \Rightarrow \) linearity cannot be used to define uncertainty tubes independently of \( \{ u_{i|k}, i = 0, 1 \ldots \} \)

The complexity of exact tubes grows rapidly with horizon hence need to consider tubes with fixed complexity e.g. polytopic tubes \( \{ X_{i|k}, i = 0, 1, \ldots \} \), where \( X_{i|k} \) has a fixed maximum number of facets/vertices

Polytopic tubes

Define the tube cross section \( X_{i|k} = \{ x : Vx \leq \alpha_{i|k} \} \) for fixed \( V \in \mathbb{R}^{n_V \times n_x} \), variable \( \alpha_{i|k}, i = 0, 1, \ldots \)

\( X_{i|k} \) is the intersection of \( n_V \) half-spaces:

\[
X_{i|k} = \bigcap_{i=1}^{n_V} \{ x : V_j x \leq (\alpha_{i|k})_j \} \\
= \{ x : Vx \leq \alpha_{i|k} \}
\]
Polytopic tubes

Assume:

(i). The system $x_{k+1} = \Phi_k x_k$, $\Phi \in \text{Co}\{\Phi^{(j)} \mid j = 1, \ldots, p\}$ is stable with joint spectral radius $\lambda^* < 1$

$$\lambda^* := \limsup_{n \to \infty} \max_{\{\Phi_1, \ldots, \Phi_n\} \in \text{Co}\{\Phi^{(j)} \mid j = 1, \ldots, p\} \times \cdots \times \text{Co}\{\Phi^{(j)}\}} \|\Phi_1 \cdots \Phi_n\|^{1/n}$$

(ii). $V$ is chosen so that $S := \{x : Vx \leq 1\}$ is $\lambda$-contractive:

$$\Phi^{(j)} S \subseteq \lambda S, \ j = 1, \ldots, p$$

for some $\lambda \in [\lambda^*, 1)$

$\lambda$-contractivity of $S$ is ensured if $S$ is computed as a positively invariant set for the dynamics $x_{k+1} = \frac{1}{\lambda} \Phi x_k$, $\Phi \in \text{Co}\{\Phi^{(j)} \mid j = 1, \ldots, p\}$

---

Polytopic tubes

To ensure that the constraints:

1. $F_h x_{i|k} + G_h u_{i|k} \leq 1$
2. $\Pr(F_s x_{i+1|k} + G_s u_{i+1|k} \leq 1) \geq p$

are satisfied at prediction times $i = 0, 1, \ldots$, we impose the constraints:

1. $X_{i|k} \subseteq \{x : \Phi^{(j)} x + B^{(j)} c_{i|k} \in X_{i+1|k}\}$
   $x_0 \subseteq X_0|k$
2. $X_{i|k} \subseteq \{x : \tilde{F}_h x + G_h c_{i|k} \leq 1\}$
   $X_{i|k} \subseteq \{x : \Pr(\tilde{F}_s \Phi x + \tilde{F}_s B c_{i|k} + G_s c_{i+1|k} \leq 1) \geq p\}$

for $i = 0, 1, \ldots$

where $\tilde{F}_h := F_h + G_h K$
$\tilde{F}_s := F_s + G_s K$

These polytopic set inclusion conditions are imposed using LP duality
Aside: LP duality

Primal: \( x^* = \arg \max_x c^T x \) subject to \( Ax \leq b \)

Dual: \( \lambda^* = \arg \min_\lambda b^T \lambda \) subject to \( A^T \lambda = c, \lambda \geq 0 \)

Theorem \( c^T x^* = b^T \lambda^* \)

Proof:

Define the Lagrangian function \( L(x, \lambda) := c^T x - \lambda^T (Ax - b) \), then

(i). \( c^T x^* = L(x^*, \lambda^*) \) since KKT conditions imply \( \lambda^*^T (Ax^* - b) = 0 \)

(ii). \( L(x^*, \lambda^*) \leq L(x^*, \lambda) \)

\[ \leq L(x, \lambda) + (x^* - x)^T \nabla_x L(x, \lambda) = L(x, \lambda) \]

since \( \lambda \geq 0 \) and \( Ax^* \leq b \)

(iii). \( L(x, \lambda) = b^T \lambda + (c - A^T \lambda)^T x \)

\[ = b^T \lambda \]

since \( \nabla_x L(x, \lambda) = 0 \)

Set inclusion conditions

Let \( S_1 = \{x : F_1 x \leq b_1\} \), \( S_2 = \{x : F_2 x \leq b_2\} \)

Theorem

\( S_1 \subseteq S_2 \) iff there exists \( H \geq 0 \) satisfying

\[ HF_1 = F_2, \quad Hb_1 \leq b_2 \]
Probabilistic set inclusion conditions

Let \( S_1 = \{ x : F_1 x \leq b_1 \} \), \( S_2 = \{ x : \Pr(F_2 x \leq b_2) \geq p \} \) for random \( F_2, b_2 \)

**Theorem**

\( S_1 \subseteq S_2 \) iff there exists a random variable \( H \geq 0 \) satisfying

\[
HF_1 = F_2, \quad \Pr(Hb_1 \leq b_2) \geq p
\]

**Proof (if):** Assume \( H \) exists, then

- for any \( x \in S_1 \), we have \( F_2 x = HF_1 x \leq Hb_1 \),
- so \( \Pr(Hb_1 \leq b_2) \geq p \) \( \Rightarrow \) \( \Pr(F_2 x \leq b_2) \geq p \) \( \Rightarrow \) \( x \in S_2 \)

**Proof (only if):**

- Assume \( S_1 \subseteq S_2 \), then \( \Pr(\mu \leq b_2) \geq p \), where, for \( i = 1, \ldots, n_F \),

\[
\mu_i := \max_x (F_2)_i x \quad \text{subject to} \quad F_1 x \leq b_1
\]

- LP duality implies

\[
\mu_i = \min_h hb_1 \quad \text{subject to} \quad hF_1 = (F_2)_i, \ h \geq 0
\]

- Define \( H \) via

\[
\left\{ \begin{array}{l}
\mu_i = h^*_i b_1, \ h^*_i F_1 = (F_2)_i, \ h^*_i \geq 0, \\
H_i := h^*_i, \ i = 1, \ldots, n_F,
\end{array} \right.
\]

then this choice of \( H \) satisfies \( \Pr(Hb_1 \leq b_2) \geq p \), \( HF_1 = F_2 \), \( H \geq 0 \)
Let $S_1 = \{x : F_1x \leq b_1\}$, $S_2 = \{x : \Pr(F_2x \leq b_2) \geq p\}$ for random $F_2, b_2$

**Theorem**

$S_1 \subseteq S_2$ iff there exists a random variable $H \geq 0$ satisfying

$$HF_1 = F_2, \quad \Pr(Hb_1 \leq b_2) \geq p$$

Note that $H$ is given explicitly in terms of $F_1, F_2$ by

$$H = F_2(F_1^TF_1)^{-1}F_1^T + PQ$$

where $QF_1 = 0$ and $P$ is a free parameter.

Therefore, given a sample set $\{F_2^{[j]} \mid j = 1, \ldots, n\}$
we can obtain a corresponding sample set $\{H^{[j]} \mid j = 1, \ldots, n\}$:

$$H^{[j]} = F_2^{[j]}(F_1^TF_1)^{-1}F_1^T + PQ, \quad j = 1, \ldots, n$$

### Set inclusion conditions

- $x_{i+1|k} \in X_{i+1|k}$ for all $x_{i|k} \in X_{i|k}$ is enforced by

  $$\alpha_{i+1|k} \geq H^{(j)}\alpha_{i|k} + VB^{(j)}c_{i|k}, \quad i = 0, 1, \ldots$$

  where $H^{(j)} = \arg\min_H H1$ subject to $HV = V_i\Phi^{(j)}$, $H \geq 0$

  $$j = 1, \ldots, p$$

- $F_hx_{i|k} + G_hu_{i|k} \leq 1$ is enforced by

  $$H_h\alpha_{i|k} + G_hc_{i|k} \leq 1, \quad i = 0, 1, \ldots$$

  where $H_h = \arg\min_H H1$ subject to $HV = \tilde{F}_h$, $H \geq 0$

- $\Pr(F_sx_{i+1|k} + G_su_{i+1|k} \leq 1) \geq p$ is enforced by

  $$\Pr(H_s\alpha_{i|k} + \tilde{F}_sbc_{i|k} + G_sc_{i+1|k} \leq 1) \geq p, \quad i = 0, 1, \ldots$$

  where $H_s = \arg\min_H H1$ subject to $HV = \tilde{F}_s\Phi$, $H \geq 0$
Terminal condition

For a prediction horizon $M$ and a control horizon $N$, with $M \geq N$:

- $\alpha_{i|k}$ is computed explicitly as a variable in the online MPC optimization for $0 \leq i \leq M$
- $\alpha_{i|k}$ satisfy $\alpha_{i+1|k} \geq H^{(j)} \alpha_{i|k}$ for $N \leq i \leq M$
- $\alpha_{i|k}$ is defined by $\alpha_{i+1|k} := \max_j \{ H^{(j)} \alpha_{i|k} \}$ for all $i \geq M$
- a terminal constraint is imposed on $\alpha_{M|k}$ so that $H_h \alpha_{i|k} \leq 1$ and $\Pr(H_h \alpha_{i|k} \leq 1) \geq p$ for all $i > M$

Terminal condition

If $S = \{ x : Vx \leq 1 \}$ is $\lambda$-contractive, then

$$\Phi^{(j)} S \subseteq \lambda S \iff \exists H \geq 0, H^{(j)} V = V \Phi^{(j)}, H^{(j)} 1 \leq \lambda 1,$$

i.e.

$$\| H^{(j)} \|_\infty \leq \lambda$$

This bound is the basis of the terminal conditions:

**Theorem**

If $\lambda \| H_h \|_\infty \| \alpha_M \|_\infty \leq 1$, then $H_h \alpha_{i|k} \leq 1$ for all $i > M$

**Proof:** $\| H^{(j)} \|_\infty \leq \lambda$ implies $\| \alpha_{i+1|k} \|_\infty \leq \lambda \| \alpha_{i|k} \|_\infty$ for all $i \geq M$

$$\Rightarrow \| H_h \alpha_{i+1|k} \|_\infty \leq \| H_h \|_\infty \| \alpha_{i+1|k} \|_\infty$$
$$\leq \lambda^{i-M+1} \| H_h \|_\infty \| \alpha_M \|_\infty$$
$$\leq \lambda^{i-M}$$
Terminal condition

If $S = \{x : Vx \leq 1\}$ is $\lambda$-contractive, then

\[
\Phi^{(j)} S \subseteq \lambda S \iff \exists H \geq 0, H^{(j)} V = V \Phi^{(j)} , H^{(j)} 1 \leq \lambda 1 , \text{i.e.}
\]

\[
\|H^{(j)}\|_\infty \leq \lambda
\]

This bound is the basis of the terminal conditions:

Theorem

If $\Pr(\lambda \|H_s\|_\infty \|\alpha_M\|_\infty \leq 1) \geq p$, then $\Pr(H_s \alpha_i | k \leq 1) \geq p$ for all $i > M$

Proof: $\|H^{(j)}\|_\infty \leq \lambda$ implies $\|\alpha_{i+1|k}\|_\infty \leq \lambda \|\alpha_i|k\|_\infty$ for all $i \geq M$

\[
\Rightarrow \|H_s \alpha_{i+1|k}\|_\infty \leq \|H_s\|_\infty \|\alpha_{i+1|k}\|_\infty \\
\leq \lambda^{i-M+1} \|H_s\|_\infty \|\alpha_{M|k}\|_\infty \\
\leq \lambda^{i-M}
\]

Receding horizon control law

Chance-constrained MPC problem formulation

**Offline:** compute $V, H_h, H_s, H^{(j)}, j = 1, \ldots, p$

**Online:** at $k = 0, 1, \ldots$:

(i). Solve $c_k^* = \arg\min_{c_k} J(x_k, c_k)$

\[
\text{s.t. } \alpha_{i+1|k} \geq H^{(j)} \alpha_i | k + VB^{(j)} c_i | k , \quad i = 0, \ldots, M - 1 \\
H_h \alpha_i | k + G_h c_i | k \leq 1 , \quad i = 0, \ldots, M \\
\Pr(H_s \alpha_i | k + \tilde{F} B c_i | k + G_s c_{i+1|k} \leq 1) \geq p , \quad i = 0, \ldots, M \\
\alpha_0 = V x_k , \quad \lambda \|H_h\|_\infty \|\alpha_M\| \leq 1 , \quad \Pr(\lambda \|H_s\|_\infty \|\alpha_M\|_\infty \leq 1) \geq p
\]

(ii). Implement $u_k = K x_k + c_0^* | k$

* Recursive feasibility: $c_{k+1} = \{c_{1|k}^*, \ldots, c_{N-1|k}^*, 0\}$ is feasible at time $k + 1$

* $u_k \to K x_k$ and $x_k \to \gamma S$ for some $\gamma > 0$ in finite time since $S = \{x : Vx \leq 1\}$ is $\lambda$-contractive hence $x = 0$ is asymptotically stable
Receding horizon control law

Sampled MPC problem formulation

Offline: compute $V$, $H_h$, $H_s^{[l]}$, $l = 1, \ldots, n$, $H^{(j)}$, $j = 1, \ldots, p$

Online: at $k = 0, 1, \ldots$

(i). Solve $c_k^* = \arg\min_{c_k} J(x_k, c_k)$

\[
\begin{align*}
\text{s.t. } & \alpha_{i+1|k} \geq H^{(j)} \alpha_{i|k} + V B^{(j)} c_{i|k}, & i = 0, \ldots, M - 1 \\
& H_h \alpha_{i|k} + G_h c_{i|k} \leq 1, & i = 0, \ldots, M \\
& H_s^{[l]} \alpha_{i|k} + \tilde{F} B^{[l]} c_{i|k} + G_s c_{i+1|k} \leq 1 & i = 0, \ldots, M \\
& \alpha_0 = V x_k, \; \lambda \|H_h\|_\infty \|\alpha_M\| \leq 1, \; \lambda \|H_s^{[l]}\|_\infty \|\alpha_M\|_\infty \leq 1, \\
& \forall l \in I, \; |I| \geq q
\end{align*}
\]

(ii). Implement $u_k = K x_k + c_{0|k}^*$

- Recursive feasibility: $c_{k+1} = \{c_{1|k}^*, \ldots, c_{N-1|k}^*, 0\}$ is feasible at time $k + 1$

- $u_k \to K x_k$ and $x_k \to \gamma S$ for some $\gamma > 0$ in finite time

hence $x = 0$ is asymptotically stable

Receding horizon control law

Summary of convergence argument:

- Recursive feasibility and the definition of the cost as an expected value (or nominal) quadratic cost implies

\[
c_{0|k}^* \to 0 \quad \text{as} \quad k \to \infty
\]

and, for any $\epsilon > 0$, there exists finite $n$ such that

\[
|c_{0|k}^*| \leq \epsilon 1 \quad \forall k \geq n
\]

- $\lambda$-contractivity of $S = \{x : V x \leq 1\}$ (i.e. $\Phi^{(j)} S \subseteq \lambda S \; \forall j$) implies

\[
V x_{k+1} \leq \max_j V (\Phi^{(j)} x_k + B^{(j)} c_{0|k}^*) \leq \lambda V x_k + \epsilon \max_j V B^{(j)} 1, \quad \text{for all} \; k \geq n
\]

- Hence $\lambda < 1$ implies

\[
\lim_{m \to \infty} V x_{n+m} \leq \epsilon \frac{1}{(1 - \lambda)} \max_j V B^{(j)} 1
\]

so, for any $\epsilon' > 0$, there exists finite $r$ such that $V x_k \leq \epsilon' 1$ for all $k \geq r$

and hence $u_k = K x_k$ for all $k \geq r$
Theorem
The sampled MPC law \( u_k = Kx_k + c^*_0|_k \) satisfies the probabilistic constraint
\[
\Pr_x(F_s x_{k+1} + G_s u_{k+1} \leq 1) \geq p
\]
with a confidence of at least \( F_{n,q - Nn_u}(p) \) at each time \( k = 0, 1, \ldots \)

Proof:
A solution of the sampled MPC optimization is feasible for
\[
\begin{align*}
\min_{c_k} J(x_k, c_k) \\
\text{subject to} \quad & F_h x_{i|k} + G_h c_{i|k} \leq 1, \quad i = 0, \ldots, M \\
& F_s (\Phi[j] x_{i|k} + B[j] c_{i|k}) + G_s c_{i+1|k} \leq 1, \quad i = 0, \ldots, M \\
& \forall j \in I \subset \{1, \ldots, n\}, |I| \geq q
\end{align*}
\]
and the solutions of this problem are feasible for the chance-constrained MPC optimization with confidence at least \( F_{n,q - Nn_u}(p) \).

Numerical example

- System matrices:
  \[
  A = A_0 + \sum_{j=1}^{3} w_j \Delta_A^{(j)}, \quad B = B_0 + \sum_{j=1}^{3} w_j \Delta_B^{(j)}
  \]
  with random \((w_1, w_2, w_3)\) defined by a centre-weighted Dirichlet distribution such that \(|w_1 + w_2 + w_3| \leq 1\), and
  \[
  A_0 = \begin{bmatrix} -1.765 & -0.866 \\ 0.677 & 0.322 \end{bmatrix}, \quad \Delta_A^{(1)} = \begin{bmatrix} -0 & 0.05 \\ -0.05 & 0 \end{bmatrix}, \quad \Delta_A^{(2)} = \begin{bmatrix} -0.01 & -0.05 \\ 0 & 0.01 \end{bmatrix}, \quad \Delta_A^{(3)} = \begin{bmatrix} -0.01 & 0 \\ 0.05 & 0.01 \end{bmatrix},
  \]
  \[
  B_0 = \begin{bmatrix} 1.939 \\ -0.505 \end{bmatrix}, \quad \Delta_B^{(1)} = \begin{bmatrix} -0.06 \\ 0.05 \end{bmatrix}, \quad \Delta_B^{(2)} = \begin{bmatrix} -0.06 \\ 0.05 \end{bmatrix}, \quad \Delta_B^{(3)} = \begin{bmatrix} 0 \end{bmatrix}
  \]

- Constraints: \( F_h = 0, G_s = 0, \)
  \[
  G_h = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad F_s = \begin{bmatrix} -1.7 & 6.0 \\ 1.7 & -6.0 \end{bmatrix}
  \]
  Probabilistic constraints required to be satisfied with probability \( p = 0.75 \)

- Cost: \( J(x_k, c_k) := \sum_{i=0}^{\infty} \mathbb{E}(\|x_{i|k}\|^2 + 100u_{i|k}^2) \)
Numerical example

100 closed loop simulations:
Stochastic MPC via sampling

Numerical example

100 closed loop simulations:
Robust MPC (all constraints imposed with $p = 1$)
Numerical example

Prediction horizons: \( N = 3, M = 5 \)

\( n = 420 \) samples taken from the probability distribution of \((A, B)\)

\( q = 332 \) gives 95\% confidence of probabilistic constraint satisfaction

Summary of results for 500 closed loop simulations

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<th>Sampled SMPC</th>
<th>Robust MPC</th>
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<tr>
<td>Mean closed loop cost</td>
<td>436.2</td>
<td>475.6</td>
</tr>
<tr>
<td>% satisfaction ( k = 1 )</td>
<td>77.2</td>
<td>100</td>
</tr>
<tr>
<td>% satisfaction ( k = 2 )</td>
<td>78.4</td>
<td>100</td>
</tr>
<tr>
<td>Average execution time (ms)</td>
<td>98.5</td>
<td>11.3</td>
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