Delay and Accessibility in Random Temporal Networks

2nd Symposium on Spatial Networks

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Accessibility in Deterministic Static and Temporal Networks

Static Networks

Defined be adjacency matrix $\mathbf{A}$. $(\mathbf{A}^n)_{ij}$ gives the number of paths of length n or less between vertices i and j.

Accessibility matrix $\mathcal{P}(n)$ in n steps is defined as follows.

$$\mathcal{P}_{ij}(n) = \begin{cases} 1 & \text{if } (\mathbf{A}^n)_{ij} > 0 \\ 0 & \text{if } (\mathbf{A}^n)_{ij} = 0 \end{cases}.$$

Temporal Networks

Defined by a sequence of adjacency matrices $\mathbf{A}_1, \ldots, \mathbf{A}_T$ over T time slots. Matrix $\mathcal{C}_T = \prod_{i=1}^{T}(\mathbf{1} + \mathbf{A}_i)$ gives the total number of *temporal paths* between any two vertices.

Accessibility matrix $\mathcal{P}(n)$ in n steps is defined as follows.

$$\mathcal{P}_{ij}(n) = \begin{cases} 1 & \text{if } (\mathcal{C}_T)_{ij} > 0 \\ 0 & \text{if } (\mathcal{C}_T)_{ij} = 0 \end{cases}.$$

Accessibility in Deterministic Static and Temporal Networks (Cont'd)

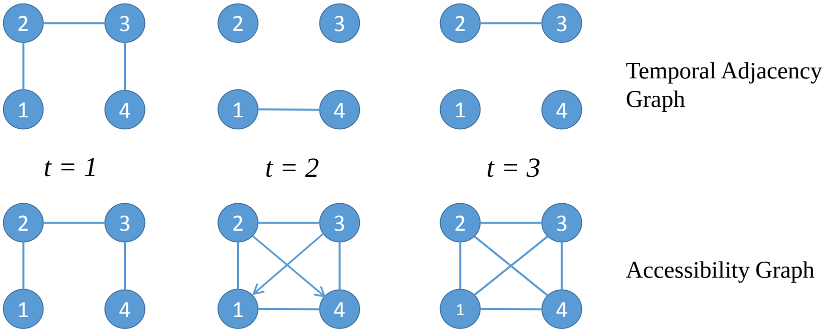

Figure: Temporal Networks

Accessibility in Random Temporal Networks: Problem Definition

- The set vertices $V = \{1, \ldots, N\}$.
- Discrete time slots $T = 1, 2, \ldots$
- Probability of an edge between i and j at time t: $p_{ij}(T)$.
- If an edge exists between i and j, $e_T(i, j) = 1$, otherwise $e_{ij}(T) = 0$.
- A temporal path between i and j by time T: $A_m^{ij}(T) = v_m^{ij}(0) \ldots v_m^{ij}(T)$ (where $v_m^{ij}(0) = i$, $v_m ij(T) = j$ and $v_m^{ij}(t) \in \{1, \ldots, N\}, \forall t \in \{1, \ldots, T-1\}$).
- All paths from i to j: $A^{ij}(T) = \{A_1^{ij}(T), \ldots, A_{M(T)}^{ij}(T)\}$.
- Temporal path from i to j with $v_m^{ij}(T-1) = \ell$: $B_m^{i\ell j}(T)$.
- $B^{i\ell j}(T) = \{B_1^{i\ell j}, \ldots, B_{M(T)}^{i\ell j}\}$.
- Enumerator of paths $1 \leqslant m \leqslant M(T) = N^{T-1}$

Accessibility in Random Temporal Networks: Problem Definition (Cont'd)

Open Path

A temporal path between i and j by time T denoted by $A_m^{ij}(T) = v_m^{ij}(0)\ldots v_m^{ij}(T)$ (where $v_m^{ij}(0) = i$, $v_m ij(T) = j$ and $v_m^{ij}(t) \in \{1,\ldots,N\}$), is defined to be *open* if for any two successive pair of *distinct* vertices $e_T\big(v_m^{ij}(t)v_m^{ij}(t+1)\big) = 1$.

Objective

Our goal is to find $P(i \overset{T}{\longleftrightarrow} j)$ (or a bound for it), where $P(i \overset{T}{\longleftrightarrow} j)$ is the probability of at least one open temporal path between vertices i and j.

Exact Method : Erdös-Rényi Random Temporal Networks

- We assume $p_{ij}(T) = p(T), \forall i, j \in V$
- We start from vertex i to visit other vertices. Any vertex u at time $t = 1$ is labeled as *visited* if $e_1(i, u) = 1$.
- The set of visited vertices at time T: $\omega(T)$.
- The set of all vertices visited from $t = 1$ to $t = T$: $W(T)$.
- A vertex is labeled as visited in time T if there exist an edge between any the vertices in $W(T - 1)$.
- $W(T + 1) = W(T) \cup \omega(T)$.

Exact Method (Cont'd)

Calculation of $P(i \overset{T}{\longleftrightarrow} j)$

We have $|\omega(1)| \sim B(N, p(1))$ and $|\omega(T)| \sim B(N, 1-(1-p(T))^{|W(T-1)|})$. Thefore we can conclude that:

$$P(|W(T)| = k) =$$

$$\sum_{\ell=0}^{k} P(|W(T-1)| = \ell)\binom{N-\ell}{k-\ell}(1-(1-p(T))^{\ell})^{k-\ell}(1-p(T))^{N-k}$$

$$\Rightarrow P(i \overset{T}{\longleftrightarrow} j) = P(j \in W(T)) =$$

$$\sum_{\ell=1}^{N} P(j \in W(T) \Big| |W(T)| = \ell) P(|W(T) = \ell) =$$

$$\frac{\ell}{N-1} P(|W(T)| = \ell)$$

Exact Method (Cont'd)

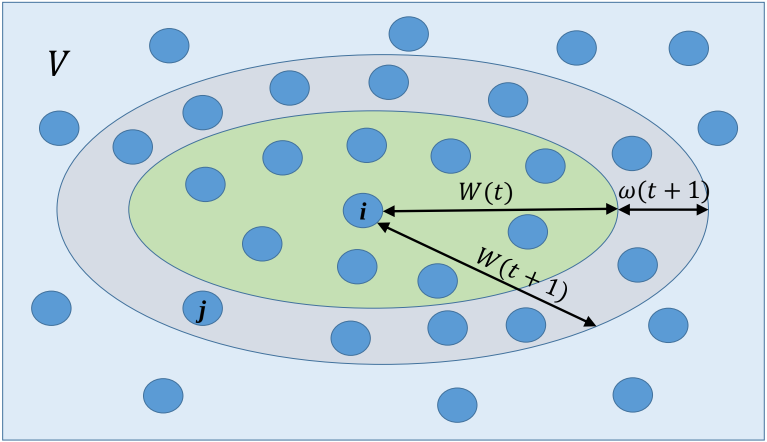

Figure: Node j has not been visited by time t. In time slot $t + 1$ node j falls into the set of nodes labeled as visited.

Upper Bound for General Random Temporal Networks

Definition

A family $\mathcal{A}$ of subsets of $K = \{1, \ldots, k\}$ is *monotone decreasing* if $A \in \mathcal{A}$ and $A^{'} \subseteq A \Rightarrow A^{'} \in \mathcal{A}$. Similarly, it is *monotone increasing* if $A \in \mathcal{A}$ and $A \subseteq A^{'} \Rightarrow A^{'} \in \mathcal{A}$.

FKG-Harris Inequality

Let $\mathcal{A}$ and $\mathcal{B}$ be two monotone increasing families of subsets of K and let $\mathcal{C}$ and $\mathcal{D}$ be two monotone decreasing families of subsets of K. Then for any real vector $p = (p_1, \ldots, p_k), 0 \leqslant p_i \leqslant p_k$

$$P_{r_p}(\mathcal{A} \cap \mathcal{B}) \geqslant P_{r_p}(\mathcal{A}).P_{r_p}(\mathcal{B})$$
$$P_{r_p}(\mathcal{C} \cap \mathcal{D}) \geqslant P_{r_p}(\mathcal{C}).P_{r_p}(\mathcal{D})$$
$$P_{r_p}(\mathcal{A} \cap \mathcal{C}) \leqslant P_{r_p}(\mathcal{A}).P_{r_p}(\mathcal{C})$$

Upper Bound (Cont'd):

Definition

$$\alpha_{ij}(T) = 1 - \prod_{\ell=1}^{N} \left(1 - \alpha_{i\ell}(T) p_{\ell j}(T+1)\right), \alpha_{ij}(1) = p_{ij}(1)$$

Theorem

$P(i \overset{T}{\longleftrightarrow} j) \leqslant \alpha_{ij}(T)$, for all $(i,j) \in V \times V$ and any positive integer $T \geqslant 1$.

Proof

Induction: $T = 1 : \alpha_{ij}(1) = p_{ij}(1) \checkmark$

$P(i \overset{T}{\longleftrightarrow} j) \leq \alpha_{ij}(T) \Rightarrow P(i \overset{T+1}{\longleftrightarrow} j) \leq \alpha_{ij}(T+1)$.

$$P(i \overset{T}{\longleftrightarrow} j) = P\left(\bigcup_{m=1}^{M(T)} A_m^{i\ell}(T)\right) \leqslant \alpha_{ij}(T) \Rightarrow$$

Upper Bound (Cont'd)

Proof (Cont'd)

$$P\Big(\bigcup_{m=1}^{M(T)} A_m^{i\ell}(T)\Big)p_{\ell j}(T+1) = P\Big(\Big[\bigcup_{m=1}^{M(T)} A_m^{i\ell}(T)\Big] \cap \{\ell j\}\Big) =$$

$$P\Big(\bigcup_{m=1}^{M(T)} (A_m^{i\ell}(T) \cap \ell j)\Big) \Rightarrow P\Big(\bigcup_{m=1}^{M(T)} B_m^{i\ell j}(T+1)\Big) \leqslant \alpha_{i\ell}(T)p_{\ell j}(T+1)$$

$$\Rightarrow P\Big(\bigcap_{m=1}^{M(T)} \overline{B}_m^{i\ell j}(T+1)\Big) \geqslant 1 - \alpha_{i\ell}p_{\ell j}(T+1)$$

$$\xRightarrow{FKG} P\Big(\bigcup_{m=1}^{M(T+1)} A_m^{ij}(T+1)\Big) = P\Big(\bigcup_{\ell=1}^{N}\Big(\bigcup_{m=1}^{M(T+1)} B_m^{i\ell j}(T+1)\Big) \leqslant$$

$$1 - \prod_{\ell=1}^{N} P\Big(\bigcap_{m=1}^{M(T)} \overline{B}_m^{i\ell j}(T+1)\Big) \leqslant 1 - \prod_{\ell=1}^{N}\big(1 - \alpha_{i\ell}(T)p_{\ell j}(T+1)\big)$$

$$= \alpha_{ij}(T+1)$$

Lower Bound I

Algorithm

- We form the the probability matrix $\mathcal{M}$ where $\mathcal{M}_{ij} = \min_t\{p_{ij}(t)\}, \forall\, t = 1, \dots, T$.
- Define a set of thresholds $S = \{p_1, \dots, p_{|S|}\}$.
- **For** $\ell = 1 : N(N-1)$

  Form G^ℓ as follows.

  $$G^\ell_{ij} = \begin{cases} 1 & \text{if } \mathcal{M}_{ij} > p_\ell \\ 0 & \text{if } \mathcal{M}_{ij} < p_\ell \end{cases}.$$

  Find the maximum clique in G^ℓ. Denote the size of clique by N_ℓ.

  Apply **Exact Method** for a network with fixed probability p_ℓ and size N_ℓ. Denote the probability with $P(\ell)$.

  End
- $P_L = \max_\ell\{P(\ell)\}$

Lower Bound I (Cont'd)

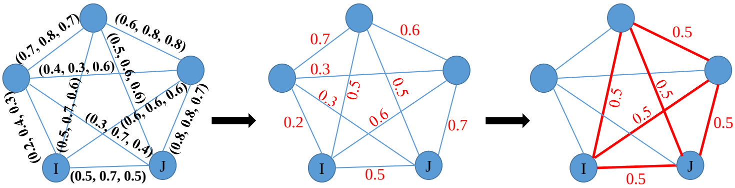

Figure: Lower Bound I: Searching for a clique based on a given threshold $p_{min} = 0.5$. The triplet on each edge represents the probabilities of links' be open in time slots $t = 1$, $t = 2$ and $t = 3$.

Lower Bound II

Edge Disjoint Paths

Fact: Any two edge disjoint paths are independent of each other.

$\Rightarrow$ Any set of disjoint paths gives a lower bound on the $P(i \overset{T}{\longleftrightarrow} j)$. If denote the set of edge disjoint paths by $R_1, \ldots, R_d$, then we have $P(i \overset{T}{\longleftrightarrow} j) \geq 1 - \prod_{k=1}^{d} P(R_k)$.

Probability of a Path

Suppose $R_k = e^k_{r_1} \ldots, e^i_{r_{L_k}}$.

$p^k_{min} = min_{l \in \{1,\ldots,L_k\}} \{\mathcal{M}_{e_{r_l} e_{r_{l+1}}}\}$.

quality of a path:

$$P(R_k) \geqslant 1 - \sum_{m=1}^{L_k - 1} \binom{T}{j} (1 - p_{min})^{T-m} p^m_{min} = f(R_k)$$

The objective would be to find a set of high quality paths between an origin I and a destination J.

Lower Bound II (Cont'd)

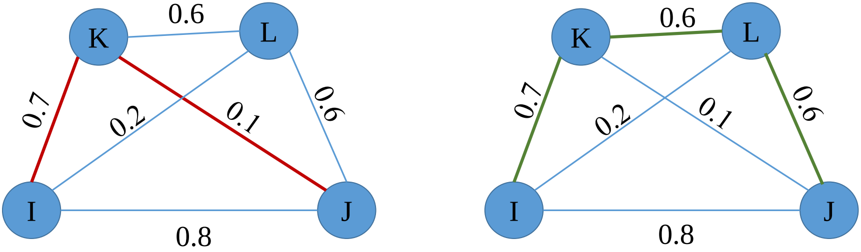

Figure: Comparing two paths $R_a = I, K, J$ and $R_b = I, K, L, J$ based on their quality, $f\big(R_a(10)\big) = 0.26$ (on the left) and $f\big(R_b(10)\big) = 0.98$ (on the right).

Quantization of Continuous Probability Distributions of ON-OFF Periods

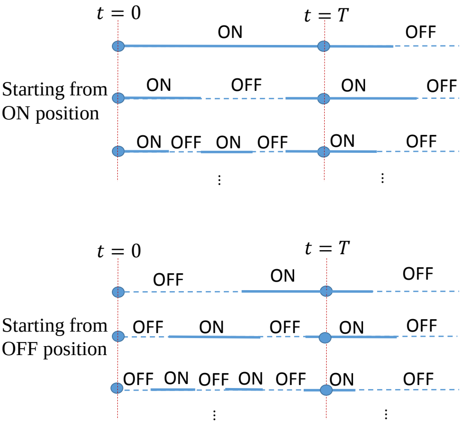

Figure: Different possibilities (events) based on starting from ON or OFF position for a given edge.

Quantization of Continuous Probability Distributions of ON-OFF Periods

Problem Formulation

If we are given the probability distributions of the ON and OFF periods (for a specific link) denoted by $f_{ON}(\tau)$ and $f_{OFF}(\tau)$ and also the probability of starting from ON position , we can obtain the probability of being in ON position (denoted by $SW = 1$, and $SW = 0$ for OFF) at time t_0 which is obtained as follows.

$$P(SW = 1) = p0 \sum_{i=0}^{\infty} \int_0^{t_0} f_{S_i^{ON}}(s)\big(1 - F_{ON}(t0 - s)\big)ds +$$

$$(1 - p_0) \sum_{i=0}^{\infty} \int_0^{t_0} f_{S_i^{OFF}}(s)\big(1 - F_{ON}(t0 - s)\big)ds$$

$$f_{S_i^{ON}}(s) = \begin{cases} 1 & \text{if } i = 0 \\ \underbrace{f_{ON} * \cdots * f_{ON}}_{i} \underbrace{* f_{OFF} \cdots * f_{OFF}}_{i} & \text{if } i > 0 \end{cases} .$$

$$f_{S_i^{OFF}}(s) = \underbrace{f_{ON} * \cdots * f_{ON}}_{i} \underbrace{* f_{OFF} \cdots * f_{OFF}}_{i+1}$$

Numerical Experiments: Synthetic Networks

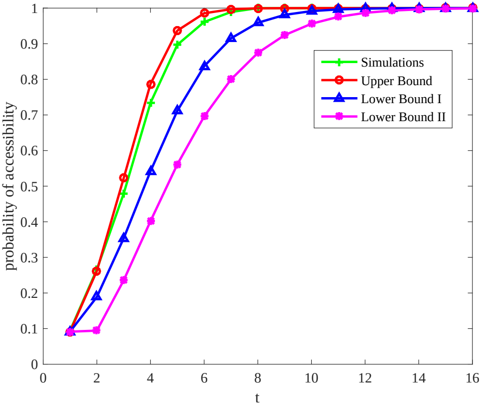

Figure: Probability of accessibility between two (randomly selected) nodes in a network with $N = 20$ nodes. The probability of each edge is uniformly selected from the range [0.05, 1] and remains constant during the observation window of 16 time slots.

Numerical Experiments: Synthetic Networks (Cont'd)

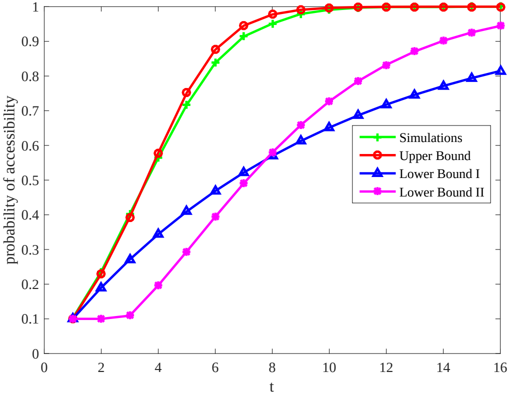

Figure: Probability of accessibility between two randomly selected nodes for the partially connected network. In this network of $N = 20$ nodes a fraction $\alpha = 0.5$ of the edges are assumed to be open with a fixed probability $p = 0.1$ and the rest of edges will not be open at any time.

Numerical Experiments: Synthetic Networks (Cont'd)

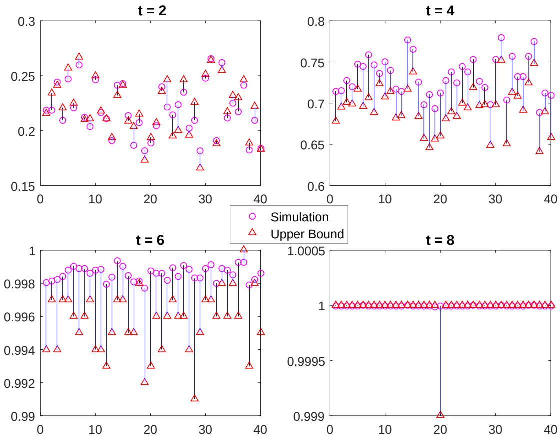

Figure: Comparison between the upper bound and Monte-Carlo simulations of the probability of accessibility for 40 randomly selected pairs in four time slots, $t = 2$, $t = 4$, $t = 6$ and $t = 8$

Numerical Experiments: Synthetic Networks (Cont'd

$$\pi(D_{i \to j}) = \frac{dP(i \xrightarrow{t} j)}{dt}$$

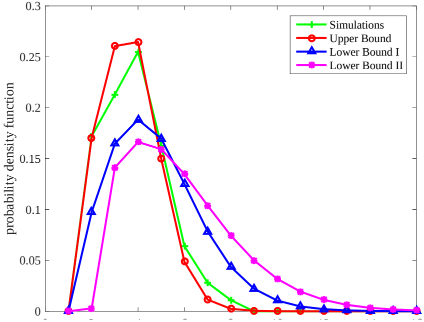

Figure: Accessibility delay distribution (for the experiment setting of Fig. 6). Such delay is defined as the elapsed time for a node j to become accessible from i for the first time (for uniform probabilities of edges).

Numerical Experiments: Synthetic Networks(Cont'd)

$$\pi(D_{i \to j}) = \frac{dP(i \xrightarrow{t} j)}{dt}$$

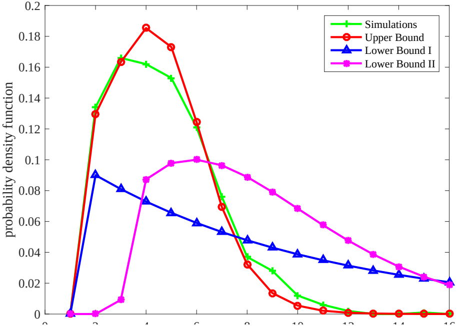

Figure: Accessibility delay distribution (for the partially connected network setting experiment in Fig. 7.)

Numerical Experiments: Synthetic Networks(Cont'd)

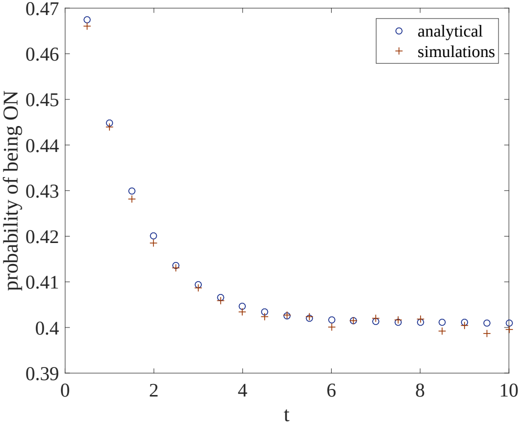

Figure: Comparing probability of an edge existing between two specific nodes (randomly selected) obtained from simulations and ON-OFF analytical model for exponentially distributed ON-OFF periods

Numerical Experiments: Real World Data

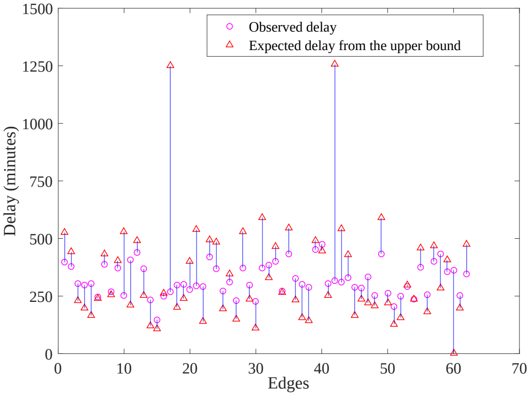

Figure: Comparing the expected delay derived from the upper-bound and estimated edge probabilities over the training phase, and the average delay from the 10 experiments. Due to the relatively small size of the dataset and possibility of unexpected events (e.g absence of a vehicle from the network) a few points of discrepancy are not surprising.

Numerical Experiments: Real World Data (Cont'd)

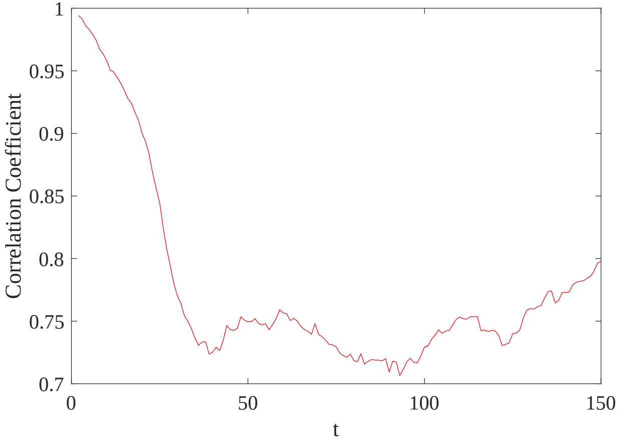

Figure: Pearson's correlation coefficient between the predicted probabilities of accessibility and the empirically estimated values.

Current Research : Accessibility in Spatio-Temporal Networks

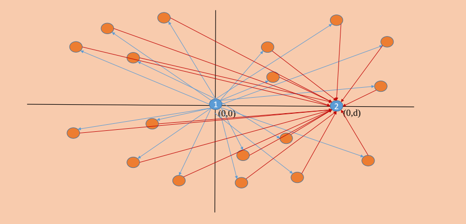

Figure: Two-hop connectivity in a random spatio-temporal network

PPP with parameter λ. We assume $p_{ij}(t) = e^{-r_{ij}^2}$

- $T = 1$: $P(1 \overset{1}{\leftrightarrow} 2) = p_{12}(1) = e^{-d^2}$
- $T = 2$: $P(1 \overset{2}{\leftrightarrow} 2) = 1 - p_{12}^2(1) exp\left[\frac{-\pi\lambda}{2} e^{-\frac{1}{2}d^2}\right]$ (D. Hedges, et al., 2017)
- $T = 3$:
  $P(1 \overset{2}{\leftrightarrow} 2) = 1 - \left[1 - P(1 \overset{2}{\leftrightarrow} 2)\right]\mathbb{E}\left[\prod_{\ell\in\Phi}(1 - \alpha_{i\ell}(2)p_{\ell j}(3))\right]$
  where $\alpha_{i\ell}(2) = P(i \overset{2}{\leftrightarrow} \ell)$.