RANDOM GRAPHS AND WIRELESS COMMUNICATION NETWORKS

Part 4: Modelling and Analysis of Ad Hoc Networks

1.5 hours

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Outline

• Applications of ad-hoc networks

• Modelling ad hoc networks
  • Random Geometric Graphs
  • Pairwise Connection function
  • Anisotropic nodes
  • Multiple Antennas

• Local Observables
  • Mean degree
  • Pair Formation
  • Degree distributions
  • Clustering coefficient

• Global Observables
  • Full connectivity
  • Boundary effects
  • K-connectivity
Ad hoc Networks

key ingredients

• Decentralized (no central BS but scalable)
• No pre-existing infrastructure “Place and Play”
• Self-configuring “on the fly”
• Multi-hop Routing (dynamic and adaptive)
  ✓ Table-driven (proactive) routing
  ✓ On-demand (reactive) routing
  ✓ Hybrid (both proactive and reactive) routing
  ✓ Hierarchical routing protocols (tree-based)
• Mobility (MANETS & VANETS)
• SmartPhone (SPANs) D2D, Bluetooth, WiFi-direct, LTE-direct
Applications of ad-hoc networks

- Standardized under: IEEE 802.15.4
  - ZigBee, WirelessHART, ISA100.11a, and MiWi
- Wireless sensor networks (WSN)
  - Environmental, Agricultural, Industrial, Military
  - Disaster relief solutions
- Building automation
  - Smart metering, Industrial control
- Internet of Things
  - Smart Cities
- Agricultural / Infrastructure / Environmental monitoring
Modelling ad hoc (random) networks

A statistical framework
Modelling ad hoc (random) networks

- Number and Location of wireless devices
  - Ad hoc, mobile, physical constraints and costs
- Multipath (fast fading)
- Shadowing (slow fading)
- Power control
  - Cooperation - signalling overheads
- MAC protocols
  - TDMA / FDMA / CDMA / SDMA ...
  - ALOHA / CSMA / CD / CA (802.11)
- Directional antennas
- Multiple antennas
- Transmission scheme (MRC / STBC)
Random geometric network
Random geometric network

Confined Random geometric networks

What is the probability of achieving a fully connected network at a given density?
The goal is to develop a theory for $P_{fc}$ that is able to provide useful analytic and physical insight which can support a variety of connectivity models, and can act as a basis for further development and analysis. To this end we will derive a general formula for $P_{fc}$ which is simple, intuitive and practical.
Pairwise Connection function

Complement of outage probability

\[ H_{ij} = \mathbb{P}[\text{SINR}_{ij} \geq q] \]

\[ \text{SINR}_{ij} = \frac{\mathbb{P}|h_{ij}|^2 g(d_{ij})}{\mathcal{N} + \gamma \mathcal{I}_j} \]

\[ g(d_{ij}) = \frac{1}{\epsilon + d_{ij}^\eta} \]

\[ \eta > 2 \]

\[ |h_{ij}|^2 \sim \exp(1) \]

\[ \gamma = 0 \]

Path loss attenuation function

Path loss exponent

Channel gain

Interference factor


Pairwise Connection function

Rayleigh Fading

\[ |h_{ij}|^2 \sim \exp(1) \]

Soft = Stochastic/Probabilistic

Hard = Deterministic
Rayleigh Fading

\[ |h_{ij}|^2 \sim \exp(1) \]

Rician Fading

\[ E_T(t) = \cos 2\pi ft \]
\[ E_R(t) = \sum_{i=0}^{N} a_i \cos(2\pi ft + \phi_i) \]
\[ E_R(t) = (a_0 + X) \cos 2\pi ft - Y \sin 2\pi ft \]
inphase and quadrature

\[ X = \sum_{i=1}^{N} a_i \cos \phi_i \]
\[ Y = \sum_{i=1}^{N} a_i \sin \phi_i \]
\[ X, Y \sim N(0, \sigma^2) \quad \text{(CLT)} \]
\[ P = (a_0 + X)^2 + Y^2 \]

Rician Pairwise Connection function

\[ f_P(x) = \frac{1}{2} \exp \left( -\frac{x + \lambda}{2} \right) I_0 \left( \sqrt{\lambda x} \right) \]
\[ \lambda = \left( a_0 / \sigma \right)^2 \]
Anisotropically radiating nodes

\[
\text{SNR} \propto G_i G_j
\]

\[
H_{ij} = \exp \left( -\frac{\beta r_{ij}^n}{G_i(\theta_j)G_j(\theta_j)} \right)
\]

**Normalization:** Antenna Tx and Rx gains

\[
\int_0^{2\pi} \int_0^\pi G(\theta) \sin \theta d\theta d\phi = 4\pi
\]

**Patch**

\[
G(\theta) = 1 + \epsilon \cos \theta
\]

**Dipole**

\[
G(\theta) = \frac{2\Gamma(\frac{3+m}{2})}{\sqrt{\pi} \Gamma(\frac{2+m}{2})} \sin^m \theta
\]

**Horn**

\[
G(\theta) = \frac{2(\lambda^2 - 1) \cos \lambda \theta}{(\lambda \sin \frac{\pi}{2\lambda} - 1)}
\]

Anisotropically radiating nodes

\[ G'(\theta) = 1 + \epsilon \cos \theta \]

\[ H_{ij} = \exp \left( -\frac{\beta r_{ij}^\eta}{G_i(\theta_j)G_j(\vartheta_j)} \right) \]

Multiple antennas

1) Single Input Single Output (SISO)

\[ H_{ij} (r) = e^{-\beta r^n} \]

2) Single Input Multiple Output (SIMO/MISO)

\[ H_{ij} (r) = \frac{\Gamma (m, \beta r^n)}{\Gamma (m)} \]

3) Multiple Input Multiple Output (MIMO-MRC) with 2 receiving and \( n \) transmitting antennas (or vice versa)

\[
H_{ij} (r) = 1 - n P (n - 1, \beta r^n) P (n + 1, \beta r^n) \\
+ (n - 1) P (n, \beta r^n)^2
\]

\( P (n, x) \) is the regularised lower incomplete gamma function

- \( n = 2, 5, 10 \)
- \( \eta = 2, 2, 6 \)
Multiple antennas

\[ H_{ij} = \mathbb{P}[\text{SNR}_{ij} \geq q] \]

SISO: \[ \text{SNR}_{ij} = \frac{\mathcal{P}|h_{ij}|^2 g(d_{ij})}{\mathcal{N}} \]
\[ |h_{ij}|^2 \sim \exp(1) \]

MIMO: \( n \times m \) channel matrix \( \mathbf{H} \)

(\text{STBC})

\[ \text{SNR}_{ij} = \frac{\mathcal{P} |\mathbf{H}|_F^2 g(d_{ij})}{\mathcal{N}} \]
\[ |\mathbf{H}|_F^2 = \sum_{k,l} |h_{kl}|^2 \]
\[ \chi^2 \text{ distributed with } 2mn \text{ dof} \]

\[ H_{ij}(r) = \frac{\Gamma(mn, \beta r^\eta)}{\Gamma(mn)} \]
Local Network Observables

The probability of some node $i$ connecting with some other node

$$H_i(r_i) = \frac{1}{V} \int _V H(r_{ij}) \, dr_j$$

Pair formation probability: The probability that 2 randomly selected nodes connect to form a pair

$$p_2 = \frac{1}{V^2} \int _{V^2} H(r_{ij}) \, dr_i \, dr_j$$

Degree distribution: The probability that node $i$ connects with exactly $k$ other nodes

$$d_i(k) = \binom{N - 1}{k} H_i^k (1 - H_i)^{N-1-k}$$

$$d_i(k) \approx \frac{\lambda_i^k}{k!} e^{-\lambda_i} \quad \lambda_i = (N - 1) H_i$$

Mean degree

$$\lambda = \int \lambda_i \, dr_i / V = (N - 1) p_2$$

$$\lambda \sim (N - 1) \pi / (\beta V) \approx \rho \pi / \beta$$
Local Network Observables

2-node correlation function

Probability that node 1 connects with node 3, given that node 1 is connected with node 2:

\[ C'(r, \eta) = \frac{\int_{\mathbb{R}^2} H_{12} H_{13} \, d\mathbf{r}_1}{\int_{\mathbb{R}^2} H_{12} \, d\mathbf{r}_1} \]

\[ C'(r, \eta) = 2^{-2/\eta} - \frac{\eta^{-2/\eta}}{8\Gamma(1 + 2/\eta)} r^2 + \mathcal{O}(r^4) \]

\( \eta = 2, 4, 6, \infty \)

Nearby nodes are less correlated for soft connectivity functions.
Global Network Observables

Global observables:
Given a graph, what is the probability of achieving full connectivity? (Erdös 1959)

A graph is fully connected if there exists at least one multi-hop path connecting every two nodes.
A cluster expansion in 3 simple steps

1) Start with the probability of two nodes being connected (or not)

\[ 1 = H_{ij} + (1 - H_{ij}) \]

2) Multiply over the complete graph to get the probability of all possible combinations giving \(2^{N(N-1)/2}\) terms

\[ 1 = \prod_{i<j} [H_{ij} + (1 - H_{ij})] = \sum_{g} \mathcal{H}_g \]

3) Group into collections of terms determined by their largest cluster

\[ 1 = \sum_{g \in G_N} \mathcal{H}_g + \sum_{g \in G_{N-1}} \mathcal{H}_g + \ldots + \sum_{g \in G_1} \mathcal{H}_g \]

\[ \underbrace{P_{fc}}_{G_{N-1}} \]

4) At high densities full connectivity is simply the complement of the probability of an isolated node.

\[ P_{fc} = 1 - \sum_{g \in G_{N-1}} \mathcal{H}_g - \ldots \]

1. Start with the probability of two nodes being connected (or not):

\[ 1 \equiv H_{ij} + (1 - H_{ij}) \]

2. Multiply over the complete graph to get the probability of all possible combinations:

\[ 1 \equiv \prod_{i<j} [H_{ij} + (1 - H_{ij})] = \sum_g \mathcal{H}_g \]

3. Group into collections of terms determined by their largest cluster:

\[ 1 = \sum_{g \in G_{N,1}} \mathcal{H}_g + \sum_{g \in G_{N,N-1}} \mathcal{H}_g + \ldots + \sum_{g \in G_{N,N}} \mathcal{H}_g \]
At high densities, *full connectivity* is the complement of an isolated node:

\[ P_{fc} = 1 - \sum_{g \in G_{N-1}} \mathcal{H}_g - \ldots \]

3. Group into collections of terms determined by their largest cluster:
Define an average over all possible configurations

\[ \langle A \rangle = \frac{1}{V^N} \int_{\mathcal{V}^N} A(r_1, r_2, \ldots, r_N) \, dr_1 dr_2 \ldots dr_N \]

\[ P_{fc} \approx 1 - \langle \sum_{g \in G_{N-1}} \mathcal{H}_g \rangle \]

1) Node \( N \) is not connected to any of the other \( N-1 \) nodes
2) Multiply by \( N \) since all nodes are identical

3) Since we are comparing pairs of nodes, \( N-2 \) integrals can be decoupled through a change of variables:

\[ x_j = r_j - r_N \]
\[ dx_j = dr_j \]

\[ = 1 - \frac{N}{V^N} \int_{\mathcal{V}^N} \prod_{j=1}^{N-1} (1 - H(r_{jN})) \, dr_1 \ldots dr_N \]

\[ = 1 - \frac{N}{V} \left( 1 - \frac{1}{V} \int_{\mathcal{V}} H(r_{1N}) \, dr_1 \right)^{N-1} \, dr_N \]
The Homogeneous case

Assuming that the network is homogeneous, implies that there are no boundaries and therefore the system is symmetric under translational transformations. This allows for a final change of variables and we are left with a single integral:

\[
P_{fc} \approx 1 - N \left( 1 - \frac{1}{V} \int_V H(r) dr \right)^{N-1} = 1 - Ne^{-\rho \int_V H(r) dr} \left[ 1 + \frac{1}{N} \left( \rho \int_V H(r) dr - \frac{\left( \rho \int_V H(r) dr \right)^2}{2} \right) + O \left( \frac{\rho^4}{N^2} \right) \right]
\]

\[e^x = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n\]

Example of a homogeneous network space: Surface of a Sphere
The Inhomogeneous case

System is not symmetric under translational transformations and so border effects become important.

Here are some simple examples:

Here are some more interesting (non-convex) examples:
Inhomogeneous problem (Boundary effects)

System is not symmetric under translational transformations

\[
P_{fc} \approx 1 - \left\langle \sum_{g \in G_{N-1}} \mathcal{H}_g \right\rangle
\]

\[
= 1 - N\left\langle \prod_{j=1}^{N-1} (1 - H_j) \right\rangle
\]

1) Node \( N \) is not connected to any of the other \( N-1 \) nodes

2) Multiply by \( N \) since all nodes are identical

3) Since we are comparing pairs of nodes, \( N-2 \) integrals can be decoupled through a change of variables:

\[
x_j = r_j - r_N
\]

\[
dx_j = dr_j
\]

For more details see Ref. 2

4) Assume that \( N \) is large, express the bracket as an exponential, and re-label node \( N \) to 2.

\[
= 1 - \frac{N}{V^N} \int_{V^N} \prod_{j=1}^{N-1} (1 - H(r_{jN})) dr_1 \ldots dr_N
\]

\[
= 1 - \frac{N}{V} \int_{\mathcal{V}} \left( 1 - \frac{1}{V} \int_{\mathcal{V}} H(r_{1N}) dr_1 \right)^{N-1} dr_N
\]

\[
= 1 - \rho \int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(r_{12}) dr_1} (1 + \mathcal{O}(N^{-1})) dr_2
\]

Observation: The mass of the pair connectedness function is in the exponent.

Conclusion: Exterior integral is maximum when interior integral is minimum.

Full connectivity is dominated by regions in the network space that are hard to connect to i.e. near the boundaries!
Example: Ad hoc network in a disk domain

\[ P_{fc} = 1 - \rho \int_{V} e^{-\rho \int_{V} H(r_{12}) dr_{1}} dr_{2} \]

**Example 1: Disk domain of radius \( R \)**

1) Use Euclidean distance between two nodes in polar coordinates:

\[ d(r_1, r_2) = \sqrt{|r_1|^2 + |r_2|^2 - 2|r_1||r_2| \cos \theta} \]

2) Set \( \eta = 2 \) and consider SISO link model. Interior integral gives connectivity mass:

\[
\int_{dR} H(r_{12}) dr_1 = \int_{0}^{R} \int_{0}^{2\pi} \left( r_1 e^{-\beta(r_1^2+r_2^2-2r_1r_2 \cos \theta)} \right) d\theta dr_1 \\
= 2\pi \int_{0}^{R} \left( r_1 I_0(2r_1r_2\beta)e^{-\beta(r_1^2+r_2^2)} \right) dr_1,
\]

3a) Taylor expand integrand around \( r_2 = 0 \) and integrate to obtain connectivity mass **away** from the boundaries:

\[
= \frac{\pi}{\beta} \left( 1 - e^{-\beta R^2} \right) + \mathcal{O}(r_2^2) \approx \frac{\pi}{\beta} \beta R^2 \gg 1
\]

3b) Use asymptotic expression of modified Bessel function of he first kind \( I_0(x) = \frac{e^x}{\sqrt{2\pi x}} (1 + \mathcal{O}(x^{-1})) \)

to obtain the connectivity mass **near** boundaries:

\[
= 2\pi \int_{0}^{R} \left( r_1 \frac{e^{2r_1r_2\beta}}{\sqrt{4\pi r_1r_2\beta}} e^{-\beta(r_1^2+r_2^2)} \right) dr_1 \\
\approx \frac{\sqrt{\pi}}{\sqrt{\beta}} \int_{0}^{R} e^{-\beta(r_1-r_2)^2} dr_1 \\
= \frac{\pi}{2\beta} \left( \text{erf} \left[ \sqrt{\beta} (R - r_2) \right] + \text{erf} \left[ \sqrt{\beta} r_2 \right] \right)
\]

4) Matching the two solutions we obtain an approximation for the connectivity mass:

\[
\approx \frac{\pi}{2\beta} \left( \text{erf} \left[ \sqrt{\beta} (R - r_2) \right] + 1 \right)
\]
Example: Ad hoc network in a disk domain

\[ \int_{d_1} H(r_{12}) dr_1 = \int_0^R \int_0^{2\pi} \left( r_1 e^{-\beta (r_1^2 + r_2^2 - 2r_1r_2 \cos \theta)} \right) d\theta dr_1 \]
\[ \approx \frac{\pi}{2\beta} \left( \text{erf} \left[ \sqrt{\beta}(R - r_2) \right] + 1 \right) \]
\[ = \frac{\pi}{2\beta} f(r_2) \]

Check approximation obtained using $\beta=1$ and $R=10$. Dots are obtained from numerical integration.

5) Further approximate $f(r)$ by a piecewise linear function:

\[ \tilde{f}(r) = \begin{cases} 
  c_1, & \text{for } 0 < r < a, \\
  c_2 - m(r - R), & \text{for } a \leq r < R 
\end{cases} \]

\[ c_1 = 2 \text{erf} \left[ \sqrt{\beta} \frac{R}{2} \right] \approx 2 \]
\[ c_2 = \text{erf} \left[ \sqrt{\beta} R \right] \approx 1 \]
\[ m = \frac{2\sqrt{\beta}}{\sqrt{\pi}} \left( 1 - e^{-\beta R^2} \right) \approx \frac{2\sqrt{\beta}}{\sqrt{\pi}} \]
\[ \beta R^2 \gg 1 \]
Example: Ad hoc network in a disk domain

6) Calculate probability of full connectivity using the piecewise linear approximation of connectivity mass:

\[
P_{fc} = 1 - \rho \int_{dR} e^{-\rho \int_{eR} H(r_{12})} dr_1 dr_2
\approx 1 - 2\pi \rho \int_{0}^{R} r e^{-\rho \frac{\pi}{2\beta} \tilde{f}(r)} dr
\]

\[
= 1 - \pi R^2 \rho e^{-\frac{\pi}{2\beta}} \left( 1 - \frac{\sqrt{\pi}}{\sqrt{\beta} R} - \frac{2\sqrt{\beta}}{\rho \sqrt{\pi} R} + O(R^{-2}) \right) - 2\pi R \sqrt{\frac{\beta}{\pi}} e^{-\frac{\pi}{2\beta} \rho} \left( 1 - \frac{\sqrt{\beta}}{\rho \sqrt{\pi} R} + O(R^{-2}) \right)
\]

\[
\tilde{f}(r) = \begin{cases} 
    c_1, & \text{for } 0 < r < a, \\
    c_2 - m(r - R), & \text{for } a \leq r < R
\end{cases}
\]

Figures: Black curves are numerical simulations. Dashed curve only includes “Area” term. Full curve is the new analytic prediction. Notice that at high densities there is excellent agreement.
A general formula for the probability of full connectivity

\[ P_{fc} = 1 - \rho \int_{V} e^{-\rho \int_{V} H(r_{12}) dr_{1}} dr_{2} \]

1) Boundary components separate and can be summed individually
2) Thus we can postulate the following general formula:

\[ P_{fc} \approx 1 - \sum_{i=0}^{d} \sum_{j_{i}} \rho^{1-i} G_{j_{i}} V_{j_{i}} e^{-\rho \omega_{j_{i}}} \int_{0}^{\infty} r^{d-1} H(r) dr \]

3) The first sum runs over objects of different co-dimension with \( i=0 \) being the volume term, and \( i=d \) being the corner terms.
4) The second sum runs over objects of equal co-dimension e.g. for a cube in \( d=3 \), we have 1 volume term, 8 faces, 12 edges, and 8 corners.
5) \( G_{j_{i}} \) is a geometric factor which is \( H(r) \) dependent and can be calculated independently for each distinct boundary component.
6) \( V_{j_{i}} \) is the volume of each object with respect to the appropriate dimension
7) \( \omega_{j_{i}} \) is the solid angle available from the corresponding object e.g. for a cube in \( d=3 \) it is simply \( 4\pi \) for the volume term, \( 2\pi \) for faces, \( \pi \) for edges, and \( \pi/2 \) for corners
8) The remaining radial integral is \( d \)-dimensional Homogeneous connectivity mass.

The simple format of this general formula emphasizes the logical decomposition of the domain into objects of different full connectivity importance. Reusable once the terms have been found for particular boundary components – a type of Universality.
What kinds of geometries can this theory analyse?

\[ P_{fc} \approx 1 - \sum_{i=0}^{d} \sum_{j_i} \rho^{1-i} G_{ji} V_{ji} e^{-\rho \omega_{ji}} \int_0^{\infty} r^{d-1} H(r) \, dr \]

- Answer: *LOTS!*
  - Limited to convex geometries
  - Complicated polyhedrons, like prisms
  - Example: a right prism in the shape of...

...a house!
Example: “house” domain (using MIMO model)

1) We consider a 2x2 MIMO pair connectedness function with $\eta = 2$:

$$ H(r) = e^{-\beta r^2} \left( \beta^2 r^4 + 2 - e^{-\beta r^2} \right) $$

2) We expect that $P_{fc}$ is a sum of the different boundary contributions:

$$ P_{fc} \approx 1 - \rho (C_1 + C_2 + E_1 + E_2 + F + U) $$

3) Start by considering the corner terms $C_1$ and $C_2$ using cylindrical coordinates:

$$ d(r_1, r_2) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_1 - \theta_2) + (z_1 - z_2)^2} $$

4) Substituting this into $H(r)$ and Taylor expanding around $r_2 = 0$ and $z_2 = 0$ (i.e. near the corner) and keeping only linear terms, we can calculate the connectivity mass:

$$ M_{H}(r_2) = \int_{0}^{\infty} \int_{0}^{\partial} \int_{0}^{\infty} r_1 H(r_{12}) \, dr_1 \, d\theta_1 \, dz_1 $$

$$ = \frac{1}{8\beta} \left( 14z_2 \vartheta + \frac{23 - \sqrt{2}}{2} \sqrt{\frac{\pi}{\beta}} \vartheta + 7\pi r_2 (\sin \theta_2 - \sin (\theta_2 - \vartheta)) \right) $$

5) We can now also calculate the exterior integral to obtain a general expression for the corners

$$ \int_{V} e^{-\rho M_H(r_2)} \, dr_2 = \int \int \int r_2 e^{-\rho \int_{V} H(r_{12}) \, dr_1} \, dr_2 \, dz_2 \, d\theta_2 $$

$$ = \frac{256\beta^3 \csc \vartheta}{343\pi^2 \rho^3 \vartheta} e^{-(\frac{23-\sqrt{2})\sqrt{\pi\rho\vartheta}}{16\beta^{3/2}}}.$$
6) We now considering the Edge terms $E_1$ and $E_2$ using cylindrical coordinates such that $r=z=0$ corresponds to the midpoint of the edge:

$$d(r_1, r_2) = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) + (z_1 - z_2)^2}$$

7) Taylor expanding around $r_2=0$ and $z_2=0$ (i.e. near the midpoint of the edge) and keeping only linear terms, we can calculate the connectivity mass:

$$M_H (r_2) = \frac{1}{4\beta} \left( \frac{23 - \sqrt{2}}{2} \sqrt{\frac{\pi}{\beta}} \vartheta + 7\pi r_2 (\sin \theta_2 - \sin (\theta_2 - \vartheta)) \right)$$

8) Note that we have assumed that $\sqrt{\beta} L \gg 1$ so that we can make the following approximations: $\exp(-\beta L^2/4) \approx 0$ and $\text{erf} \left( L\sqrt{\beta}/2 \right) \approx 1$

9) We can now also calculate the exterior integral to obtain a general expression for the edges

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{0}^{\vartheta} \int_{0}^{\infty} r_2 e^{-\rho M_H(r_2)} dr_2 d\theta_2 dz_2 = \frac{16L \beta^2 \csc \vartheta}{49 \pi^2 \rho^2} e^{- \frac{(23-\sqrt{2})\sqrt{\pi} \rho \vartheta}{8\beta^{3/2}}}$$
10) Having already considered the corners and edges, we can treat the **Surface** term independently and so the house domain with surface area: \( S = 2B + ph \) is topologically equivalent to a sphere with surface area: \( S = 4\pi R^2 \).

11) We use spherical coordinates \( d(r_1, r_2) = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos \theta} \) and calculate the connectivity mass by Taylor expanding around \( r_2 = R \) (i.e. near the surface):

\[
M_H (r_2) = 2\pi \int_0^R \int_0^\pi r_1^2 \sin \theta \ H (r_{12}) \ d\theta \ dr_1 \\
= \frac{\pi}{4\beta} \left( \frac{23 - \sqrt{2}}{2} \sqrt{\frac{\pi}{\beta}} + 14 (R - r_2) \right)
\]

12) We can now also calculate the exterior integral:

\[
2\pi \int_0^R \int_0^\pi r_2^2 \sin \theta \ e^{-\rho M_H (r_2)} \ d\theta \ dr_2 = \frac{8\beta R^2}{7\rho} e^{-\frac{(23-\sqrt{2})\pi^{3/2}}{8\beta^{3/2} \rho}}
\]
Finally we consider the **Volume** term of a sphere with equal volume as the house domain:

We use spherical coordinates and calculate the connectivity mass by Taylor expanding around \( r_2 = 0 \) (i.e. away from the surface):

\[
M_H (r_2) = 2\pi \int_0^\infty \int_0^\pi r_1^2 \sin \theta \ H (r_{12}) \ d\theta dr_1
\]

\[
= \frac{(23 - \sqrt{2})}{4} \left( \frac{\pi}{\beta} \right)^{\frac{3}{2}}.
\]

The exterior integral gives:

\[
2\pi \int_0^R \int_0^\pi r_2^2 \sin \theta \ e^{-\rho M_H(r_2)} d\theta dr_2 = V e^{-\frac{(23-\sqrt{2})\pi^{3/2}}{4\beta^{3/2}}}.
\]
Example: “house” domain (using MIMO model)

For a 2x2 MIMO pair connectedness function with $\eta = 2$:

$$H(r) = e^{-\beta r^2} \left( \beta^2 r^4 + 2 - e^{-\beta r^2} \right)$$

we have that $P_{fc}$ is a sum of the different boundary contributions:

$$P_{fc} \approx 1 - \rho \left( C_1 + C_2 + E_1 + E_2 + F + U \right)$$

\[
C_1 = 6 \frac{512 \beta^3}{343 \pi^3 \rho^3} e^{-\frac{(23-\sqrt{2})\rho}{32} \left( \frac{\pi}{\beta} \right)^{3/2}}
\]

\[
C_2 = 4 \frac{1024 \sqrt{2} \beta^3}{1029 \pi^3 \rho^3} e^{-\frac{(23-\sqrt{2})3\rho}{64} \left( \frac{\pi}{\beta} \right)^{3/2}}
\]

\[
E_1 = L \left( 9 + 2\sqrt{2} \right) \frac{16 \beta^2}{49 \pi^2 \rho^2} e^{-\frac{(23-\sqrt{2})\rho}{16} \left( \frac{\pi}{\beta} \right)^{3/2}}
\]

\[
E_2 = 2L \frac{16 \sqrt{2} \beta^2}{49 \pi^2 \rho^2} e^{-\frac{(23-\sqrt{2})3\rho}{32} \left( \frac{\pi}{\beta} \right)^{3/2}}
\]

\[
F = \frac{2 \beta S}{7 \pi \rho} e^{-\frac{(23-\sqrt{2})^{3/2} \rho}{8 \beta^{3/2}}}
\]

\[
U = Ve^{-\frac{(23-\sqrt{2})^{3/2} \rho}{4 \beta^{3/2}}}
\]
$P_{fc}(k)$ \textbf{k-connectivity:} network remains fully connected if any $k-1$ nodes are randomly removed
K-connectivity - Reliability

The probability of network having minimum degree $k$

$$P_{md}(k) = \langle \prod_{i=1}^{N} P(\text{degree}(r_i) \geq k) \rangle = \langle \prod_{i=1}^{N} (1 - D_i(k-1)) \rangle$$

$$\approx [1 - \langle D_i(k-1) \rangle]^N.$$  

(1 - probability node has degree at most k-1)^N

$$D_i(k) = \sum_{m=0}^{k} d_i(m)$$

$$d_i(k) \approx \frac{\lambda_i^k}{k!} e^{-\lambda_i}$$

$$\lambda_i = (N-1) H_i$$

$$H_i(r_i) = \frac{1}{V} \int_{V} H(r_{ij}) dr_j$$

$$M_H(r_i) = V H_i$$

K-connectivity - Reliability

Q. What about $P_{fc}(k)$?

A. $P_{md}(k)$ and $P_{fc}(k)$ have the same asymptotic distribution.

Since 2-connectivity implies 1-connectivity

$$P_{fc}(2) = P_{fc}(1) - X(1)$$

$X(1) =$ probability of obtaining a fully connected network which is not 2-connected.

At high densities, a fully connected network which is not 2-connected will typically contain a single node which is of degree 1.

$$X(1) \approx \left( \sum_{i=1}^{N} \sum_{j \neq i} H_{ij} \prod_{k \neq j \neq i} (1 - H_{ik}) \right)$$

$$= \rho^2 \int_{\mathcal{V}} M_H(r_1)e^{-\rho M_H(r_1)}dr_1,$$

Repeating this argument $k$ times:

$$P_{fc}(k) = P_{fc}(1) - \sum_{m=1}^{k-1} X(m),$$

$$X(m) = \frac{\rho^{m+1}}{m!} \int_{\mathcal{V}} M_H^m(r_1)e^{-\rho M_H(r_1)}dr_1$$
K-connectivity - Reliability

The probability of network having minimum degree $k$

$$P_{md}(k) = \left( \prod_{i=1}^{N} P(\text{degree}(r_i) \geq k) \right) = \left( \prod_{i=1}^{N} (1 - D_i(k-1)) \right)$$

$$\approx [1 - \langle D_i(k-1) \rangle]^N.$$ 

$$P_{fc}(k) \approx \left[ 1 - \sum_{m=0}^{k-1} \frac{\rho^m}{m!} \frac{1}{V} \int_{\mathcal{V}} M_{H}^{m}(r_i) e^{-\rho M_{H}(r_i)} dr_i \right]^N$$

$$D_i(k) = \sum_{m=0}^{k} d_i(m)$$

$$d_i(k) \approx \frac{\lambda_i^k}{k!} e^{-\lambda_i}$$

$$\lambda_i = (N - 1) H_i$$

$$H_i(r_i) = \frac{1}{V} \int_{\mathcal{V}} H(r_{ij}) dr_j$$

$$M_H(r_i) = VH_i$$
Example: The keyhole setup (non-convex)

\[ X = \text{the probability of a bridging link between the two sub-domains} \]

\[ = \text{the complement of the probability of no bridging link between the two sub-domains} \]

\[ X = 1 - \left\langle \prod_{i=1}^{N_A} \prod_{j=1}^{N_B} (1 - \chi_{ij} H_{ij}) \right\rangle_B \right|_A \]
Example: The keyhole setup (non-convex)

\[ X = 1 - \left( \prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \left( 1 - \chi_{ij} H_{ij} \right) \right)_{\mathcal{B}} \cdot \mathcal{A} \]

Assumption: all integrals separate out (independence)

\[ X \approx 1 - (1 - \langle \chi_{ij} H_{ij} \rangle_{\mathcal{B}} \cdot \mathcal{A})^{N_A N_B} \]

\[ = 1 - \exp \left( -\rho_A \rho_B \int_{\mathcal{A}} \int_{\mathcal{B}} \chi_{ij} H_{ij} d\mathbf{b}_j d\mathbf{a}_i \right) \]

\[ X \approx 1 - \exp \left( -\rho_A \rho_B \frac{\sqrt{\pi} w}{2 \beta^{3/2}} \right) \]

Plotted below using dashed curves

Clearly this was a bad assumption since connections through the keyhole are far from independent.
Example: The keyhole setup (non-convex)

“A system is said to present quenched disorder when some parameters are random variables which do not evolve with time - they are quenched or frozen. It is opposite to annealed disorder, where the random variables are allowed to evolve themselves.”

Notice how the LoS connectivity ‘cones’ overlap (correlated). We must average over each region separately.

\[
X = 1 - \left( \prod_{i=1}^{N_A} \prod_{j=1}^{N_B} (1 - \chi_{ij} H_{ij}) \right)^{N_A} \\
= 1 - \left( \left( 1 - \langle \chi_{ij} H_{ij} \rangle^B \right)^{N_B} \right)^{N_A} \\
= 1 - \left( e^{-N_B \langle \chi_{ij} H_{ij} \rangle^B} \right)^{N_A} \\
= 1 - \left( \frac{1}{V_A} \int_{A} e^{-\rho B} \int_{B} \chi_{ij} H_{ij} \, db_i \, da_i \right)^{N_A}
\]

Plotted below using solid curves
Summary

• Applications of ad-hoc networks
• Modelling ad hoc networks
  • Random Geometric Graphs
  • Pairwise Connection function
  • Anisotropic nodes
  • Multiple Antennas
• Local Observables
  • Mean degree
  • Pair Formation
  • Degree distributions
  • Clustering coefficient
• Global Observables
  • Full connectivity
  • Boundary effects
  • K-connectivity
References


