DYNAMICAL SYSTEMS, VOL. 18, NO. 3, 2003, 191–200

# Asymptotic angular stability in non-linear systems: rotation numbers and winding numbers

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*Abstract*. The asymptotic angular stability of a dynamical system may be quantified by its rotation number or its winding number. These two quantities are shown to result from different assumptions, made about the flow generating the Poincaré map which results from the sequence of homeomorphisms in $S^1$. An ergodic theorem of existence a.s. of the rotation number for non-linear systems is given. The advantages and disadvantages of both the rotation and winding numbers are discussed. Numerical calculations of the distribution of rotation number and winding number arising from different initial conditions are presented for three different chaotic maps.

*Received 2 July 2002; accepted 21 May 2003*

## 1. Introduction

In Ruffino (2000) the concept of rotation number was introduced for a composition of stationary random homeomorphisms which preserve orientation in $S^1$, hence a wider concept than the definition of Ruelle (1985) for linear systems (product of random matrices). An ergodic theorem of existence almost surely (a.s.) of this rotation number is proven there for these non-linear systems. The consistency of the concept was proved (see Ruffino 2000) by showing that this definition is adequate for application to continuous linear processes in $\mathbb{R}^2$ which have been discretely sampled in time. A random Nyquist theorem was stated (see, e.g. Papoulis 1984, Oppenheim and Schafer 1989) for linear processes in $\mathbb{R}^2$: the rotation number (normalized by $1/T$) of the discrete sampled process with period $T$ converges almost surely to the rotation number of the original continuous process when $T$ goes to zero. In simpler words, the stroboscopic observation of the system can retrace its original frequency a.s. if the frequency of observation is high enough. In this paper we deal with the open question arising in Ruffino (2000) concerning the application of our concept

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*Dynamical Systems* ISSN 1468–9367 print/ISSN 1468–9375 online © 2003 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/1468936031000155927

to non-linear maps in the plane with an invariant probability measure. In particular we investigate a number of chaotic systems.

The following provides a mathematical description of the problem. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a local diffeomorphism with $\mu$ a Borel invariant probability measure on $\mathbb{R}^2$ for the map $F$. It is well known that in this case the asymptotic radial stability is determined by the Lyapunov exponents of the product of the derivatives $dF(x)$ of $F$ along the trajectory of an initial point $p$:

$$dF^{(n)}(p) = dF(F^{(n-1)}(p)) \circ \ldots dF(p). \tag{1}$$

The existence $\mu$-a.s. of these characteristic exponents is the content of the celebrated multiplicative ergodic theorem, due originally to Oseledets (1968). This theorem corresponds to a stochastic linear algebra for the product of random matrices. For a detailed description of the applications of this result to random dynamical systems we refer the reader to Arnold (1998) and the references therein, in particular we mention the classical article by Ruelle (1979).

In this paper we study the asymptotic angular behaviour, which provides the counterpart of the multiplicative ergodic theorem. This asymptotic angular behaviour, which is induced by a sequence of linear operators (derivatives of $F$), may be quantified either by its '*rotation number*' or its '*winding number*'. In this paper, we develop a general theory of rotation numbers and winding numbers for a sequence of homeomorphisms in the circle $S^1$. While the winding number is more intuitive and likely to suffice for most applications in physics or applied mathematics, there are some cases where it may not be the correct quantity to measure. We clarify the differences between the rotation number and winding number using simple examples and discuss their advantages and disadvantages.

This paper is organized as follows: in sections 2 and 3 we define the rotation number and winding number for a sequence of homeomorphisms. In section 4 we state the main results from the theory of rotation numbers presented in Ruffino (2000). At the end of this section we compare our definition with that given in Ruelle (1985). In section 5 we present several examples, some of them with numerical simulations to illustrate the physical meaning of both the rotation and winding numbers.

## 2. The rotation number

We summarize the classical definition of rotation number for a single homeomorphism in the circle. We use Homeo($S^1$) to denote the space of homeomorphisms in the circle $S^1$, Homeo$^+$ ($S^1$) for the group of orientation preserving maps and Homeo$^-$($S^1$) for those which reverse the orientation.

We recall the classical definition of the rotation number of a fixed homeomorphism $f \in$ Homeo$^+$ ($S^1$). Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of $f$, i.e. if $\pi : \mathbb{R} \to S^1$ is the canonical projection $\pi(x) = \exp\{2\pi i x\}$ then $f \circ \pi = \pi \circ F$. We consider the argument of a complex number such that $\arg(\pi(x)) = x \pmod 1$. One easily sees the following facts: (a) $F$ is an increasing function; (b) if $F$ is a lift, then $F + k$, with $k \in \mathbb{Z}$, is another lift; (c) if $Id$ denotes the identity in $\mathbb{R}$ then $F - Id$ is periodic with period 1. The rotation number $\rho(f)$ of $f$ is the asymptotic average velocity of the evolution of the dynamics of a lift $F$ in the space state $\mathbb{R}$, i.e.

$$\rho(f) = \lim_{n\to\infty} \frac{F^{(n)}(x) - x}{n} \pmod 1, \tag{2}$$

where $F^{(n)}$ is the iteration of $F$ $n$-times. It is well known that the limit always exists and is independent of the starting point $x \in \mathbb{R}$. For these and other properties see Nitecki (1971) for example.

The rotation number for a sequence of elements in $\mathrm{Homeo}^+$ is defined, analogously, as the asymptotic average velocity of the evolution of the dynamics of a lift $F$ in the universal covering space (see section 4).

## 3. The oriented winding number

Let $f_1, f_2, \ldots : S^1 \to S^1$ be a sequence of homeomorphisms in the circle (here we consider a deterministic sequence of maps, no randomness is assumed). Given a fixed initial point $p \in S^1$, one of the most natural ways of calculating the average rotation of the orbit of $p$ around $S^1$ is to construct a monotone non-decreasing sequence of real numbers which provides the angle of the orbit at each step $n \in \mathbb{N}$. That is, let us define inductively the sequence $(\theta_n^p)_{n\geq 1}$ by $\theta_1^p = \arg(p)$ and $\theta_n^p \geq \theta_{n-1}^p$ such that $\pi(\theta_n^p) = f_n \circ \ldots f_1(p)$.

The anti-clockwise winding number is then defined by the average rate of increase of this ordered, lifted to $\mathbb{R}$, orbit of $p$:

$$w_{f_1,f_2,\ldots}(p) = \lim_{t\to\infty} \frac{\theta_n^p - \theta_1^p}{n},$$

when the limit exists. It is known (see Ruffino 2000, Proposition 2.1) that the winding number generalizes the classical definition of rotation number in the case of a constant sequence of orientation preserving homeomorphism $f: S^1 \to S^1$ that is, if $f_n \equiv f$ for all $n \in \mathbb{N}$, then, the rotation number $\rho(f)$ of equation (2) equals the winding number $w_f(p)$ for all $p \in S^1$ (in this case we simplify the notation to $w_f$). Example 1 in section 5 shows that for a non-constant sequence these two concepts may differ.

The next theorem states some other properties of the winding number for the iteration of a single homeomorphism $f$ as above.

**Theorem 1**. *Winding number satisfies the following properties*:

(a) *If* $f \in \mathrm{Homeo}^-(S^1)$ *then there are two distinct points* $a, b \in S^1$ *such that* $w_f(a) = w_f(b) = 0$ *and for all* $p \in S^1 \backslash \{a, b\}$ *we have* $w_f(p) = 1/2$.
(b) *If* $f \in \mathrm{Homeo}^-(S^1)$ *and* $\varphi \in \mathrm{Homeo}\ (S^1)$ *then* $w_{\varphi^{-1}\circ f\circ\varphi}(p) = w_f(\varphi^{-1}(p))$.
(c) *If* $f \in \mathrm{Homeo}^+\ (S^1)$ *then* $w_{\varphi^{-1}\circ f\circ\varphi} = \pm w_f$ *according to* $\varphi \in \mathrm{Homeo}^{\pm}(S^1)$.
(d) *If* $f, g \in \mathrm{Homeo}^+\ (S^1)$ *commute then* $w_{f\circ g} = w_f + w_g \pmod 1$.

**Proof.** For (a), note that a lift of an orientation reversing homeomorphism is a continuous monotone decreasing function on $\mathbb{R}$, hence, in the interval [0,1] it must cross the *Id* (mod 1) exactly twice (and at different points), say at the values corresponding to the angle of the points $a$ and $b$. Once they are fixed points of $f$ we have $w_f(a) = w_f(b) = 0$. These two points disconnect the circle $S^1$ into two open arcs, say $U$ and $V$ (see figure 1). The lift of $f$ also shows that the dynamic of $f$ is simply flipping points from the disconnected open arcs $U$ to $V$ and vice versa. Hence, for a point $p \in S^1 \backslash \{a, b\}$, say in $U$, after $2n$ iterations of $f$ we have $\theta_{2n}^p = n + \delta_n$ where $\delta_n$ is bounded by the length of $U$. Hence $w_f(p) = 1/2$. Item (b) is a direct corollary of (a).

Item (c) follows from the facts that if $f \in \mathrm{Homeo}^+\ (S^1)$ then $w_f(p) = \rho(f)$ for every point $p \in S^1$ and the invariance of the rotation number $\rho(f)$ by conjugacy

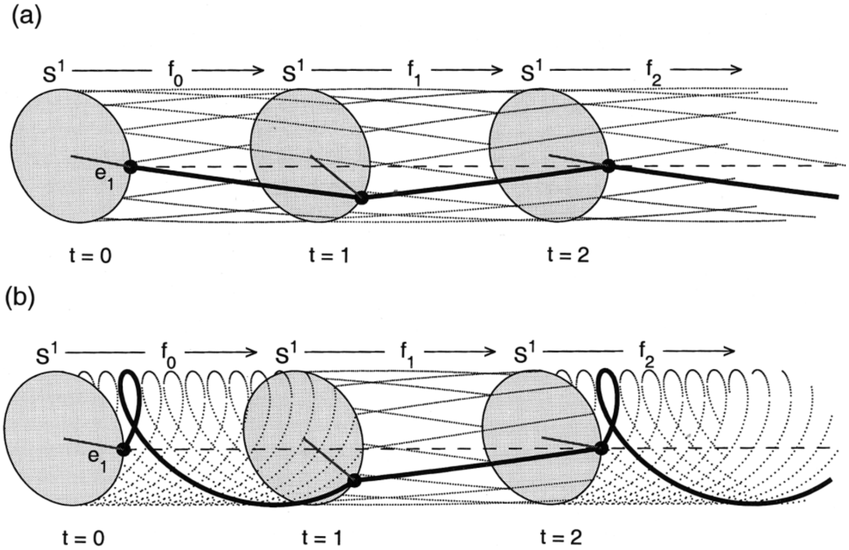

Figure 1. Stroboscopic sampling of flow on the cylinder (or torus) yields the Poincaré maps given by the sequence of homeomorphisms in $S^1$.

(see Ruffino 2000, Lemma 4.1 and remark afterwards). Item (d) follows by the same property of rotation numbers of a single homeomorphism. □

The advantage of the winding number compared with the rotation number for sequences of homeomorphisms (see section 4) is, besides its simplicity and intuitive appeal, the fact that it makes sense for general homeomorphisms, even if they do not preserve orientation. On the other hand, the disadvantage is the fact that in general the winding number depends strongly on the initial point $p$. Moreover, though intuitive, in the case of a reversing orientation homeomorphism, a flipping dynamics (which gives a winding number equal to 1/2 except in two points) does not correspond properly to any physical rotation.

## 4. Theory of rotation numbers

In this section we recall the main theoretical results concerning rotation numbers as introduced in Ruffino (2000). Let $f_1, f_2, \ldots : S^1 \to S^1$ be a sequence in Homeo$^+$ $(S^1)$. Let $F_1, F_2, \ldots : \mathbb{R} \to \mathbb{R}$ be the corresponding lifts such that $F_n(0) \in (-1/2, 1/2]$. The rotation number of this sequence of homeomorphisms is

$$\rho(f_1, f_2, \ldots) = \lim_{n\to\infty} \frac{F_n \circ \ldots F_1(x) - x}{n} \pmod 1, \tag{3}$$

when the limit exists. This obviously extends the classical definition of rotation number for a constant sequence $f_n \equiv f$ for all $n \in \mathbb{N}$. Moreover, the rotation number, in contrast to the winding number, does not depend on the initial point $x = \arg(p)$ for $p \in S^1$ (see Ruffino 2000, Proposition 3.1).

We use a geometrical illustration (see figure 1) to demonstrate the dynamical difference between the rotation number and winding number for a sequence of homeomorphisms in $Homeo^+(S^1)$. Essentially, note that the sequence of homeomorphisms represents the Poincaré map of a (time dependent) flow which may be reconstructed according to a particular rule: the rotation number employs one rule while the winding number uses a different one.

Suppose, for instance, that the sequence of homeomorphisms $(f_n)_{n\geq 1}$ is such that $f_n$ is a rotation by $\pi/6$ if $n$ is odd and $f_n$ is a rotation by $-\pi/6$ if $n$ is even. The rotation number (see figure 1a) is obtained by employing the rule that the flow between consecutive integers carries $e_1$ to the interval $(-1/2,1/2]$. Alternatively (see figure 1b) the winding number is obtained by using the rule that the trajectory of the starting point rotates anti-clockwise (independently of what happens to the other trajectories). One easily verifies that in this example, the rotation number is $\rho = 0$ and the winding number is $\omega = 1/2$.

The next two theorems summarize the main properties of rotation number. To establish a nomenclature, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta: \Omega \to \Omega$ be an ergodic transformation. Given a random homeomorphism $f: \Omega \to \mathrm{Homeo}^+(S^1)$ with distribution $\mu$ in $\mathrm{Homeo}^+(S^1)$, let $\nu$ be the invariant measure in $S^1$ for the action of a sequence of elements i.d.d. with the same distribution of $f$, that is, $\nu$ is invariant by convolution $\mu * \nu = \nu$. Let $F(\omega, x)$ be the lift of $f(\omega, s)$ and consider $\beta(\omega, s) \in (-1/2, 3/2]$ the function which measures the angular displacement induced by $f(\omega, s)$, that is, $\beta(\omega, s)$ is the unique function in this interval such that

$$F(\omega, x) = x + \beta(\omega, \pi(x)).$$

Now, considering the sequence of random stationary homeomorphisms $(f \circ \theta^{n-1})_{n\geq 1}$, we have the following ergodic theorem of existence.

**Theorem 2 (ergodic theorem for rotation number).** *The rotation number $\rho(f, \theta)$ of the sequence $(f \circ \theta^{n-1})_{n\geq 1}$ exists and is constant $\mathbb{P}$-a. s. Precisely*

$$\rho(f, \theta) = \mathbb{E}\left[\int_{S_1} \beta(\omega, s)\, d\nu(s)\right] \quad \mathbb{P} - a.s.$$

Note that ergodicity of $\nu$ is not relevant in this case since the rotation number, if it exists, is constant in $S^1$. The next theorem shows that the concept of rotation number proposed in (3) (in contrast to the winding number) makes sense from a physical perspective. The classical version of this theorem is known as the Nyquist theorem which states that: given a real signal (function) $s(t)$, $t \in \mathbb{R}$, if $f_0$ is the maximum frequency of its Fourier spectrum, then, the whole signal can be retraced if we sample in time this signal at a frequency greater than $2f_0$. This frequency is called the Nyquist rate (see., e.g. Papoulis 1984, Oppenheim and Schafer 1989). We shall state the theorem for the stochastic case; for an alternative proof of the particular case of deterministic systems or for the general case of random dynamical systems (whose proofs use the same technique), see Ruffino (2000, Theorems 5.1 and 5.2).

Consider the following Stratonovich stochastic linear system in $\mathbb{R}^2$:

$$dx_t = Ax_t\, dt + \sum_{i=1}^{m} B^i x_t \circ dW_t^i, \tag{4}$$

where $A$, $B^1, \ldots, B^m$ are $2 \times 2$-matrices, $(W_t^1, \ldots, W_t^m)_{t\geq 0}$ are $m$-independent linear Brownian motions with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\varphi(\omega, t)$ be the stochastic linear solution flow of the equation above. It is well known, see, e.g. Ruffino (2002), that the rotation number of $\varphi$ is given by

$$\rho(\varphi) = \int_{S^1} f(s)\, d\mu(s) \quad a.s.,$$

where $\mu$ is the invariant probability measure of the induced process in $S^1$ and, given $v \in S^1$ such that $(s, v)$ are orthogonal with positive orientation:

$$f(s) = \left\langle \left( A + \frac{1}{2}\sum_{i=1}^{m} (B^i)^2 \right) s, v \right\rangle - \sum_{i=1}^{m} \langle B^i s, s\rangle \langle B^i s, v\rangle.$$

In this situation, the stochastic Nyquist theorem states that the rotation number of the discrete sampled in time at intervals $T$ of the trajectories of any point $0 \neq x \in \mathbb{R}^2$ converges to the rotation number of the continuous system up to a parameter $1/T$ which normalizes the angular velocity. We recall the co-cycle property that $\varphi(\omega, nT) = \varphi(\theta_{(n-1)T}(\omega), T) \circ \cdots \varphi(\theta_T(\omega), T) \circ \varphi(\omega, T)$ where $\theta_s : \Omega \to \Omega$ is the ergodic transformation given by the shift in the noise.

**Theorem 3 (stochastic Nyquist theorem).** *The rotation number for the sampled (in time) system $\rho(\varphi(T, \omega), \theta_T)$ of the sequence of random matrices $\{\varphi(T,\theta)_{T(n-1)}(\omega)\}_{n \geq 1}$ satisfies*

$$\lim_{T \to 0} \frac{1}{T} \rho(\varphi(T, \omega), \theta_T) = \rho(\varphi) \quad a.s.$$

### 4.1. *Comments on Ruelle's rotation number*

Ruelle (1985) proposed a definition of rotation number for the particular case of a product of a sequence of random elements in the semi-simple Lie group $G = Sl^+(2, \mathbb{R})$. His method uses the intrinsic property of the linear action of the matrices (polar decomposition), it involves lifting the matrices to the covering space $\tilde{G}$ of $Sl^+(2, \mathbb{R})$ with projection $p : \tilde{G} \to Sl^+(2, \mathbb{R})$. Our concept differs from Ruelle's method by the fact that we lift the whole dynamical system from $S^1$ to its universal covering space, while Ruelle lifts only the group which acts in $S^1$ to its covering group (still linear). This is the key point which allows one to extend the definition of rotation number to non-linear dynamics in $S^1$.

We point out that there is an ambiguity in Ruelle's definition in Theorem 1 (Ruelle 1985, p. 110): consider for instance (using his notation) the probability space $M = \{a, b\}$ with $\mathbb{P}\{a\} = \mathbb{P}\{b\} = 1/2$ and $f: M \to M$ the flipping on $M$. If $p(T(a)) = Id$ and $p(T(b)) = -Id$, one can assign $\Theta(T(a)) = 0$ and $\Theta(\mathrm{T}(b)) = 3/2$, which leads to rotation number 3/4, which obviously does not agree with the physical meaning of the rotation number, which in this case should be 1/2. This ambiguity disappears if one states rules to lift from $G$ to $\tilde{G}$ (analogous to our rules at the beginning of section 4). We emphasize that any physically interesting definition of rotation number should be consistent with the discretization of continuous linear systems in the sense of Theorem 3 above. Another interesting point to distinguish our definition is that, when applied to maps in a parallelizable two-dimensional manifold (though it also depends on the trivialization of the tangent bundle) it does not depend on the isotropy to the identity, in contrast to Ruelle's definition (see Ruelle 1985).

## 5. Asymptotic angular stability in non-linear systems

In this section we consider a local diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ which has an (ergodic) invariant probability measure $\mu$ in $\mathbb{R}^2$ (more generally, this could be a local diffeomorphism on a locally parallelizable two-dimensional manifold). The asymptotic angular behaviour of the product of the matrices given by the derivative of $f$ along a trajectory is given by the rotation number of the action of these matrices in the circle $S^1$. That is, $(\mathbb{R}^2, \mu)$ plays the role of the probability space $(\Omega, \mathbb{P})$, $f$ plays the role of the ergodic transformation $\theta$, and the homeomorphisms are the induced action of the derivative $df$ along the trajectory $f^{(n)}(x)$ of a certain element $x \in \mathbb{R}^2$. The composition of homeomorphisms is given by the product of the random matrices:

$$df^{(n)}(x) = df(f^{(n-1)}(x)) \circ \ldots \circ df(p).$$

In what follows we show some examples which illustrate this asymptotic analysis both in terms of rotation number and in terms of winding number. The first example shows how one can be mislead by the winding numbers, note that: once some points move in $S^1$, one tends to think that there is an 'average non-zero rotation' (i.e. rotation number); but in fact, there is no global rotation, in other words, this phenomenon can be interpreted as local angular deformations in a ring, but the ring does not rotate as a whole.

**Example 1**. Consider the map

$$f(x, y) = (x^3 - 2x + xy,\ y).$$

The unique period-2 orbit is given by the points (1, 0) and (−1, 0). The linerization of $f$ at these points are given by

$$df_{(1,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$df_{(-1,0)} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

In both of these points the derivatives preserve the direction of $e_1$, hence the rotation number is $\rho(f) = 0$. For this initial direction, the winding number $\omega(f)(e_1)$ also vanishes. On the other hand, for any point $(x, y)$ with $y \neq 0$ one easily sees that the winding number is $\omega(f)((x, y)) = 1/2$.

This example shows that, in general, not only does the winding number differ from rotation number, but also that the winding number depends on the initial direction one chooses in the tangent space of the initial condition.

In the next examples we calculate numerically the rotation number and the winding number for some well-known chaotic systems. The procedure for this calculation consists of taking an initial condition, $p$, and computing $10^3$ iterations of the map to guarantee that this point (if not already in support of the ergodic invariant measure) will converge to the chaotic attractor. The calculation of the rotation number starts after these iterations. Similarly, for the winding number, we consider an initial condition, $p$, with the initial direction given by $e_1$. This process is repeated for a number of different initial conditions to obtain a probability density function of estimates for both the rotation and winding numbers.

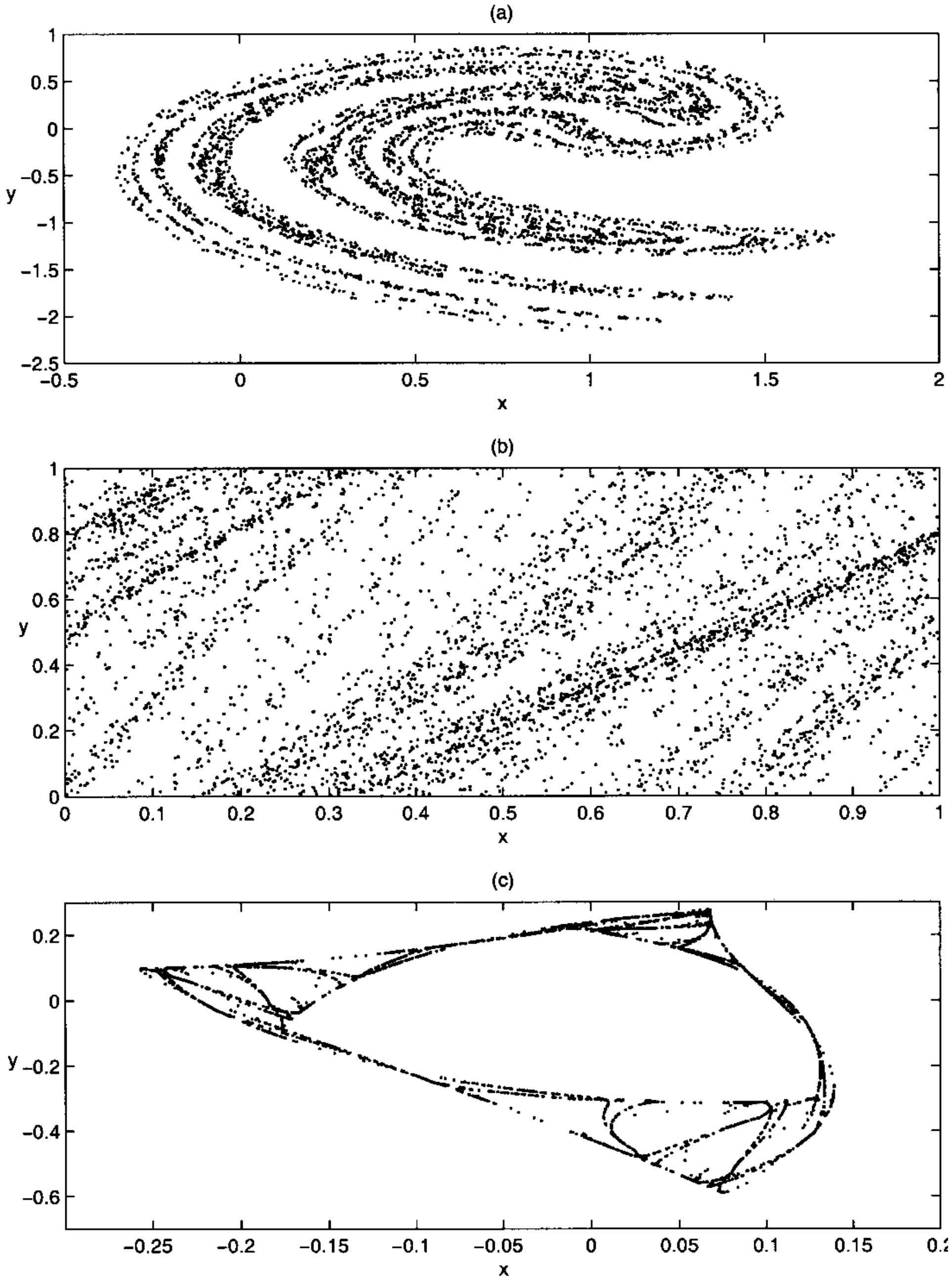

Figure 2. Attractors generated by two-dimensional chaotic maps: (a) Ikeda map, (b) Sinai map and (c) Tinkerbell map.

**Example 2.** Ikeda (1979) investigated the plane-wave model of a bistable ring cavity and noticed that it exhibited period doubling cascades to chaos. Hammel *et al.* (1985) extracted a complex difference equation relating the field amplitude at the $(i + 1)$th cavity pass to that of a round trip earlier. The amplitude $x$ and phase $y$, corresponding to the real and imaginary parts of the field are related by

$$f(x, y) = [1 + \mu(x \cos \theta - y \sin \theta), \quad \mu(x \sin \theta - y \cos \theta)] \tag{5}$$

where $\theta = \mathrm{a} - \mathrm{b}/(x^2 + y^2 + 1)$. This map is chaotic for $a = 0.4$, $b = 6$ and $\mu = 0.9$ and yields the attractor shown in figure 2a. The convergence of the estimates of $\rho(k)$ and $\omega(k)$ as a function of iteration $k$ is illustrated in figure 3a and b and their distribution at the maximum iteration $k = 2^{15}$ is shown in figure 3c and d. Note that neither $\rho(k)$ nor $\omega(k)$ is normally distributed. We describe these distributions for $k = 2^{15}$ by giving the minimum, median and maximum values along with the mean and standard deviation in tables 1 and 2. The mean values of the distribution of rotation and winding numbers are $\langle\rho\rangle = -0.29$ and $\langle\omega\rangle = 4.27$, respectively.

**Example 3.** The Sinai map (see Sinai 1972) is given by

$$f(x, y) = [(x + y + a \cos 2\pi y) \bmod 1, \quad (x + 2y) \bmod 1]. \tag{6}$$

A value of $a = 0.1$ yields a chaotic system; the system state space is shown in figure 2*b*. The mean values of the distribution of rotation and winding numbers are $\langle\rho\rangle = 0.00$ and $\langle\omega\rangle = 3.02$, respectively (see tables 1 and 2).

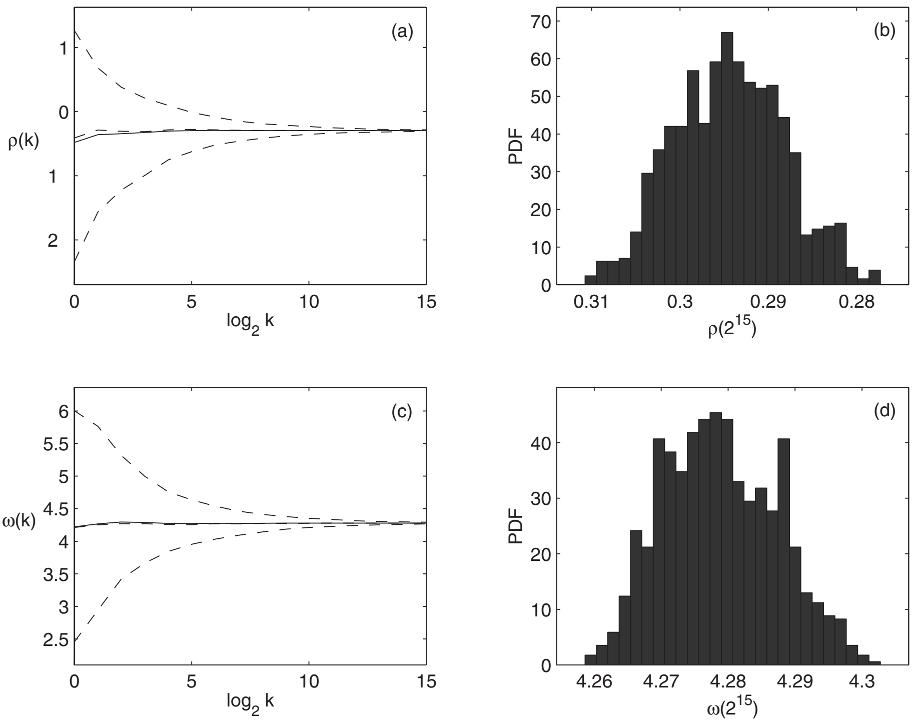

Figure 3. Rotation dynamics for the Ikeda map: (a) convergence of $\rho(k)$ with iteration $k$, (b) PDF of $\rho(k)$ for $k = 2^{15}$, (c) convergence of $\omega(k)$ with $k$ and (d) PDF of $\omega(k)$ for $k = 2^{15}$.

Table 1. Rotation numbers $\rho(k)$ for $k = 2^{15}$ iterations.

| System | Mean | Std | Min | Med | Max |
|---|---|---|---|---|---|
| Ikeda | −0.2944 | 0.0062 | −0.3106 | −0.2944 | −0.2774 |
| Sinai | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| Tinkerbell | 1.9400 | 0.0049 | 1.9223 | 1.9400 | 1.9574 |

Table 2. Winding numbers $\omega(k)$ for $k = 2^{15}$ iterations.

| System | Mean | Std | Min | Med | Max |
|---|---|---|---|---|---|
| Ikeda | 4.2788 | 0.0084 | 4.2588 | 4.2782 | 4.3026 |
| Sinai | 3.0240 | 0.0109 | 2.9905 | 3.0237 | 3.0504 |
| Tinkerbell | 2.3091 | 0.0059 | 2.2905 | 2.3088 | 2.3259 |

**Example 4.** The Tinkerbell map (see Alligood *et al.* 1996) is given by

$$f(x, y) = (x^2 - y^2 + ax + by, \quad 2xy + cx + dy) \tag{7}$$

where $a = -0.3$, $b = -0.6$, $c = 2$ and $d = -0.27$. The Tinkerbell attractor is shown in figure 2c. The mean values of the distribution of rotation and winding numbers are $\langle\rho\rangle = 1.94$ and $\langle\omega\rangle = 2.31$, respectively (see tables 1 and 2).

## Acknowledgements

This research was supported by EPSRC grant GR/N0264 and CNPq grant No. 300670/95-8.

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